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BOLTZMANN TRANSFORMATION APPROACH TO SIMULATE A TWO PHASE RADIAL DIFFUSIVITY MODEL FOR TIGHT OIL RESERVOIRS

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BY

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Declaration

I hereby declare that, this thesis is the result of my own original research and that no part of it has been submitted to any institution or organization anywhere for the award of a degree. All inclusion for the work of others has been duly acknowledged.

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Abstract

Tight oil is petroleum that accumulates in relatively impermeable reservoir rocks, often shale or tight sandstones. Globally, tight oil resources provide significant amount of petroleum for the world's energy needs. The flow behavior of tight oil in unconventional reservoirs are described by peculiar complexities that presents a challenging task in finding immediate solutions for reservoir engineers. It is therefore critical to implement approaches that solve such problems without loosing vital information of the flow phenomenon. This study demonstrates a general concept to explain the behavior of tight oil in unconventional reservoirs. In this study, an investigation into the application of similarity transformations for the analysis of complex unconventional reservoirs exhibiting two phase phenomena during transient radial flow is done. The similarity transformation is carried out with the Boltzmann variable. The techniques adopted in the transformation process aids in converting highly nonlinear partial-differential equations (PDEs) governing the two phase flow phenomenon, to nonlinear ordinary differential equations (ODEs). The resulting ODEs, consequently simplify the determination of the reservoir performance and avoid the tedious calculation ingrained in solving the original PDEs. From a theoretical point of view, the successful conversion of the highly nonlinear PDEs to ODEs permits the derivation of saturation and pressure equations as unique functions of the Boltzmann variable, which in turn, guarantees the expression of saturation as a unique function of pressure. Further research is carried out to investigate the constant gas-oil ratio (GOR) that is typically observed in some hydraulically fractured tight oil reservoirs during constant pressure two-phase production. The similarity transformation approach sets up a foundation to develop an analytical solution to the model adopted in this study. The analytical solution yielding from this work is used to obtain similar forms to well-known equations (flow rate and cumulative production) for single phase flow, which enhance our understanding of multiphase flow behavior.

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Dedication

I dedicate this work to the Almighty God for His continuous grace and my loving parents for all the support and encouragement.

Contents

List of Tables

List of Figures

List of Abbreviations and Symbols

Abbreviations

PDE	Partial Differential Equation
ODE	Ordinary Differential Equation
IMPES	Implicit Pressure Explicit Saturation
GOR	Gas-oil ratio

Symbols

A	Cross-sectional area to flow
B	Formation volume factor
H	Pay-zone Height or thickness
N_p	Cumulative oil
R_s	Solution gas oil ratio
S	Saturation
c	Compressibility
f	Fractional flow
k	Permeability
k_r	Relative permeability
k_r^*	End point relative permeability
m	Normalized two-phase pseudopressure
n	Relative permeability exponent

p	Pressure
q	Flow rate
t	Time
r	Spatial variable

Greek symbols used

μ	Viscosity
η	Hydraulic diffusivity
ρ	Fluid density
ξ	Boltzmann variable
ϕ	Porosity
∞	Infinity

Subscripts and superscripts

e	Reservoir extent
bp	Bubble point
g	Gas
i	Initial
o	Oil
w	Wellbore

Chapter 1

Introduction

1.1 Background of Study

The decline in conventional hydrocarbon resources coupled with the increase in energy demand has encouraged the development of unconventional resources. The production of oil from conventional resources has peaked and is currently on a terminal, long-run global decline. The plateau in conventional oil resources and the corresponding increase in demand for fossil fuels have triggered world markets to respond with higher oil prices. The petroleum industry is approaching the end of easily accessible, relatively homogeneous oils, and several researchers claim that, the era of cheap oil may also be ending (Gordon, 2012).

Tight oils of unconventional resources provide significant amount of petroleum for the world's energy needs. Production of tight oil comes from very low permeability rocks that must be stimulated using hydraulic fracturing mechanisms to create sufficient permeability for matured oil and/or natural gas liquids to flow at economic rates. The low permeability of tight oil reservoirs require production with large pressure drawdowns. This pressure drawdown is large enough that, the flowing pressure drops below the bubble point pressure of the in-situ liquid hence, causing the evolution of dissolved gases.

Mathematical models have been used widely to analyze the conventional reservoirs that are in existence today. However, the development of such models for unconventional reservoirs present peculiar complexities. The analysis and understanding of the factors that affect the performance of these unconventional reservoirs are critical for their efficient exploitation. The non-linearities associated with the two-phase flow, typical of tight oil reservoirs, present a challenging task in finding solutions to such models (Tabatabaie and Pooladi-Darvish, 2016).

The implementation of similarity variable theory in the context of the analysis of flow behavior in tight oil reservoirs provides one of several approaches to finding solutions to reservoir flow problems. This study develops and analyzes the applicability of similarity solutions to two phase flow in tight oil reservoirs.

1.2 Problem Statement

Reservoir flow simulation provides a reasonable approach to describe reservoir behavior, however, the equations mimic highly non-linear and complex phenomena which make their simulations tedious and computationally expensive. These complexities associated with the PDEs describing the two phase flow prevalent in tight oil unconventional reservoirs present a challenging task in finding immediate solutions for reservoir engineers. The dilemma in this setback necessitates approaches to solving such problems without loosing vital information of the flow phenomenon; a practice worthwhile to consider.

1.3 Objectives of the study

To resolve this problem, the following objectives are considered:

1. to obtain the similarity transforms of the governing equations describing the reservoir behavior.

2. to determine the fundamental physics (pressure and saturation distribution) by similarity and numerical approaches.

3. to determine the behavior of the gas-oil ratio in tight oil reservoirs.

4. to derive an analytical solution under a prescribed assumption.

1.4 Outline of Methodology

The method adopted in this study, explores the applicability of similarity transformation to transient two phase flow. The transformation is implemented with the Boltzmann variable to facilitate the conversion of the highly nonlinear partial-differential equations (PDEs) governing flow through porous media to nonlinear ordinary-differential equations (ODEs). The numerical simulation of the reservoir behavior is carried out by a finite difference method based on an Implicit-Pressure-Explicit-Saturation scheme defined on sets of hypothetical data. An analytic case is also be developed under limiting assumptions to derive solutions which are compared with proposed solutions in literature. A further investigation is carried out to study the behavior of the gas-oil ratio under a constant pressure production scenario at the sandface.

1.5 Justification of the Study

The process of building and maintaining robust, reliable model of oil fields is often time-consuming and expensive to execute. Hence, there is the need to adopt fast and efficient means to carry out simulation studies. These simulation studies are performed to offer preliminary and appraisal information about the behavior of a reservoir. This information is used by oil and gas companies in the development of new fields. Also, reservoir simulations are carried out in developed fields where production forecasts are needed to help make investment decisions. The use of similarity approximations makes the acquisition of information much handy under huge time constraints. Lowering the time for executing and analyzing a model is of utmost importance in the petroleum industry.

1.6 Scope of Work

This study expounds the basic theories and principles necessary to describe fluid flow in porous media and the techniques adopted in solving flow problems encountered in unconventional hydrocarbon resources. An investigation is carried out through the development of a reservoir model for tight oil in unconventional reservoirs which is analysed by implementing a similarity transformation as well as a numerical simulation. The results which includes profiles and plots are used to explain the practicality of the methods adopted in this study. A comparison of the semi-analytic solution and full numerical solution spanning from this study offers a platform to justify the application of similarity transformation to multiphase flow in tight oil reservoirs.

1.7 Limitations

The limitations to this work span from the assumptions made to arrive at the conclusions. These assumptions limit the analyses of this work to near ideal conditions. However, the insight obtained from this study offer a basis to consider the practicality of similarity solutions to describe hydrocarbon reservoir behavior.

1.8 Thesis organization

This study is divided into four main parts. In order to obtain an understanding of this study, Chapter two explains some of the critical background knowledge required in simulation studies of hydrocarbon reservoirs. The model building blocks such as the Law of Mass Conservation, Darcy's law and others are expounded. The implementation of similarity variable theory on partial differential equations is explored. A brief account is made about the finite difference method (Implicit-Pressure-Explicit-Saturation scheme) carried out in this study.

The third chapter explains the underlining mathematical model(PDEs) for the multiphase radial flow in tight oil reservoirs and the assumptions made to carry out the main work. Here, the similarity transformation of the model, that is, from PDEs into ODEs are resolved with the use of the Boltzmann variable. These resulting ODEs are solved numerically to obtain a semi-analytic similarity solution. This is quickly followed by a full numerical approach to solve the original PDEs by a finite difference method(IMPES). Finally, the study develops an analytical solution to the two phase radial flow of hydrocarbons yielding form the similarity transformation under a limiting assumption.

Chapter four details the analysis of a hypothetical base case which is implemented by the semi-analytic similarity and the numerical approaches. The plots from the similarity and numerical simulations are compared and explained. The comparison of the two solutions provide a means to justify the applicability of the similarity approach. This study is finally capped with conclusions and recommendation for future works in chapter five.

Chapter 2

Literature Review

2.1 Introduction

The major purpose of this chapter is to present a quick overview of the basics of reservoir engineering concepts and laws. The theories are well documented in the books of Aziz and Settari (1979), Peaceman (1977) and Zimmermann (1993). These books have been of great help in forming the backbone of this chapter. Brief descriptions of reservoir rock and fluid characteristics, rock/fluid interaction which strongly affect the multiphase flow behavior through porous media is reviewed. These petrophysical properties are well explained by Engler (2010).

2.2 Definition of Unconventional Reservoirs

Literature presents no formal definition of unconventional hydrocarbon resources, despite the fact that unconventional resources are the most vast and active petroleum systems in some parts of the world, typically Northern America. Some researchers define unconventional resources, emphasizing purely on a permeability threshold ($< 0.1md$). (Cander, 2012)

On the other hand, others prefer a definition based on the interpretation of petroleum systems and have concluded that, unconventional resources are predominantly continuous or basin centered and lack traditional trap mechanisms. Some researchers have also restricted the phrase to the fluid type that accumulates in the reservoir. There exist many shale and tight sand systems that have gas, wet gas, and oil fairways and all are considered unconventional. Notwithstanding, heavy oil and oil sands are also unconventional resources and several of such deposits are in reservoirs with permeability exceeding $500md$. (Cander, 2012)

Thus, these definitions only conclude that, unconventional resources encompasses both low and high permeability reservoirs with both low and high viscosity fluids. Nevertheless, these facts do not account for all phases of petroleum existing in the several types of reservoirs within several types of petroleum systems.(Cander, 2012)

In this study, a graphical definition that assimilates properties of both the rocks and fluid contents, is used to set up a fundamental definition. This is illustrated in Figure (2.1). Here, petroleum reservoirs are classified by using a graph of

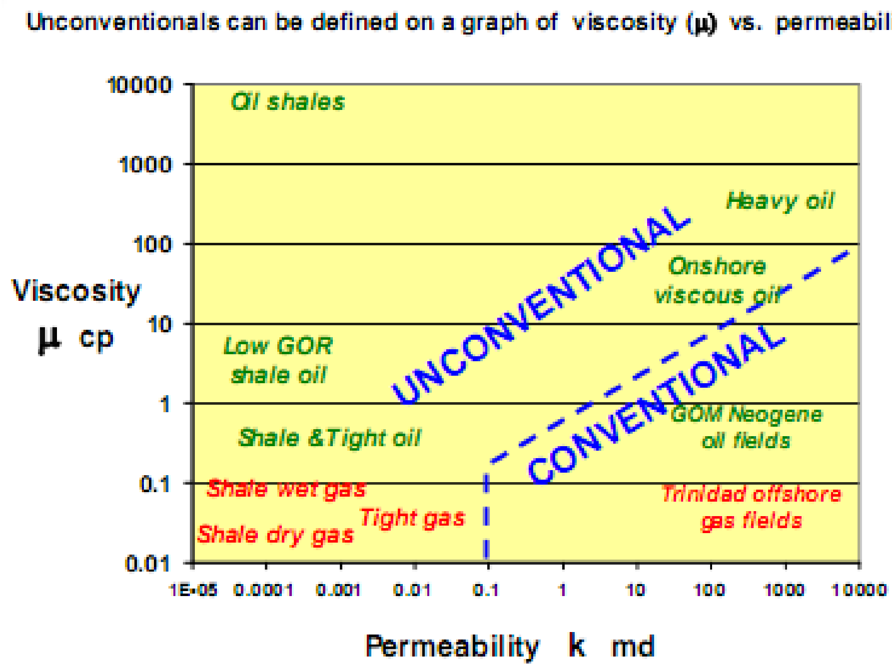


Figure 2.1: Defining unconventional resources based on fluid viscosity and rock permeability,(Cander, 2012)

viscosity versus permeability (both in log scale). By this means, conventional resources are distinguishable on the lower right quadrant of the graph, regardless of fluid phase. Unconventional resources, on the other hand, occur outside this quadrant due to a low ratio of permeability to viscosity.

Hence, unconventional reservoirs can be defined as hydrocarbon reservoirs whose permeability-viscosity ratio requires the use of technology to alter the rock permeability or the fluid viscosity in order to produce the hydrocarbon at economically viable rates. The graphical definition accommodates and delineates tight gas, tight oil, shale gas, shale oil, heavy oil, coal bed methane, and even offshore reservoirs with low permeability-viscosity ratios. (Cander, 2012)

2.2.1 Tight Oil Reservoirs

Tight oil is petroleum found in relatively impermeable reservoir rock, often shale or tight sandstones. The production of tight oil from these very low permeability rocks must be stimulated using hydraulic fracturing to create sufficient permeability to allow the mature oil and/or natural gas liquids to flow at economically viable rates (Schlumberger, 2016).

Globally, tight oil resources provide significant amount of petroleum for the world's energy needs. Table (2.1) shows estimates of technically recoverable volumes of tight oil associated with shale formations, made by the U.S. Energy Information Administration (2013).

Country	Technically recoverable volume(billion barrels)
Russia	75
United States	48 to 58
China	32
Argentina	27
Libya	26
Venezuela	13
Mexico	13
Pakistan	9
Canada	9
Indonesia	8
World Total	335 to 345

Table 2.1: Technically recoverable volume of petroleum from tight oil reservoirs in billion barrels (U.S. Energy Information Administration, 2013)

2.3 Petro-physical Properties of Porous Media

Petrophysics is the study of the physical and chemical properties of rocks and their contained fluids. The petrophysical properties of a hydrocarbon system are broadly classified into rock and fluid properties. Petrophysical properties are very important for the petroleum industry because they serve as the vital source of information required to fully understand the mechanism of rock-fluid interaction as well as determine the economic viability of hydrocarbon-bearing reservoirs. In the following, brief descriptions are given regarding some vital rock and fluid properties that are necessary to investigate hydrocarbon systems.

2.3.1 Rock Properties

Porosity

Porosity is an important rock property because it is a measure of the potential storage capacity or volume for hydrocarbons. It is defined as the ratio of pore volume to bulk volume of a rock sample which is expressed as:

$$\phi = \frac{V_p}{V_b} = \frac{\text{Pore Volume of rock}}{\text{Bulk Volume of rock}} \tag{2.1}$$

It is within these pore spaces that the oil, gas and/or water reside. Therefore a primary application of porosity is to quantify the storage capacity of the rock, and subsequently define the volume of hydrocarbons available to be produced. (Engler, 2010)

Permeability

Permeability is a measure of the ability of a porous media to transmit fluids. It is a critical property in characterizing the flow capacity of a rock sample. The unit of measurement is the *darcy*, named after the French scientist who discovered the phenomenon. It is commonly denoted as k. Absolute permeability(k) defines the ability of the porous media to transmit a particular kind of fluid under single phase conditions. Effective permeability(k_i) defines permeability of a given phase when more than one phase is present in the porous medium (Engler, 2010). Relative permeability(k_{ri}) is the ratio of the effective permeability for a particular fluid to a base or absolute permeability of the rock(Engler, 2010). Relative permeability is expressed as:

$$k_{ri} = \frac{k_i}{k} \tag{2.2}$$

where; i refers to the fluid phases (oil, gas or water).

2.3.2 Fluid Properties

Saturation

Saturation is an explicit measure of the fluid content of the porous rock. It therefore directly influences the hydrocarbon storage capacity of the reservoir. Saturation is defined as the ratio of a fluid volume to the pore volume of a porous media (Engler, 2010). For a typical hydrocarbon system the total fluid saturation is always 1; that is, $S_o + S_g + S_w = 1$.

where, it is considered that; $V_p = V_o + V_g + V_w$. Hence;

$$S_i = \frac{V_i}{V_p} = \frac{\text{Volume of oil}}{\text{Volume of pore}} \tag{2.3}$$

where; i refers to the fluid phases (oil, gas or water).

Fluid Compressibility

Fluid compressibility is defined as the fractional volumetric change of a given mass per pressure change under isothermal conditions (constant temperature). Mathematically, the coefficient of isothermal compressibility is defined as:

$$c = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \tag{2.4}$$

where V is volume and p is pressure. The subscript is used to denote that the partial differentiation is to be taken assuming constant temperature (isothermal conditions). Water, which usually occur in many hydrocarbon systems is considered as incompressible (in most cases) or slightly compressible. The compressibility of oil is dependent on the in-situ pressure existing in the system. In some cases, pressures is higher than bubble point pressure allow oils and associated solution gases to be treated as slightly compressible.(Kamyabi, 2014)

Fluid Viscosity

Fluid viscosity describes the resistance of fluids to shear stress. It is a proportional factor between the shear stress acting on a fluid and rate of deformation over time (Kamyabi, 2014). It is generally expressed as:

$$\mu = \frac{\tau}{u_x} \tag{2.5}$$

where μ represents the absolute viscosity of fluid. τ and u_x are the shear stress and rate of deformation respectively.

Formation Volume Factor

The formation volume factor is the ratio of the volume of a fluid phase at reservoir (in-situ) conditions to the volume at stock tank (surface) conditions (Engler, 2010). This factor accounts for the changes in volume of the formation fluids as it moves from reservoir conditions to surface conditions. This factor is used to convert the flow rate of fluids (at stock tank conditions) to reservoir conditions (Engler, 2010). It also enables the calculation of fluid density and it is defined as:

$$B_i = \frac{V_{ir}(\text{at reservoir conditions})}{V_{is}(\text{at surface conditions})} \tag{2.6}$$

Fluid Mobility

Fluid mobility(λ) of a fluid phase is the ease associated with the displacement of the fluid by another fluid through a porous medium It is expressed as the ratio of the relative permeability to the viscosity of a fluid phase (Engler, 2010). This property is therefore dependent on both the rock and the fluid properties. It is expressed as:

$$\lambda_i = \frac{k_{ri}}{\mu_i} \tag{2.7}$$

where i represents the phase type.

2.4 Fluid Flow in Porous Media

A porous medium refers to a rock formation or material that contains void spaces occupied by one or more fluid phases (gas, water, oil, etc.) and a solid matrix. Transient flow of a fluid through a porous medium is governed by certain types of partial differential equations known as diffusivity equations. These equations are derived through a combination of the conservation of mass equation, Darcy's law and an equation that describes the manner in which fluid is stored inside a porous rock under a set of assumptions (Engler, 2010).

2.4.1 Flow equations in radial coordinates

In order to develop a complete governing equation that applies to transient problems, the mathematical expression of the principle of conservation of mass is applied. The physics associated with fluid flow towards a well is a vital area in petroleum engineering, in which case it is more convenient to use cylindrical (radial) coordinates, rather than Cartesian coordinates (Engler, 2010). To derive the proper form of the diffusion equation in radial coordinates, fluid flow is considered towards (or away from) a vertical well, in a radially-symmetric manner. The fluids flow is a radial manner through a curved surface area, A, given by $A(r) = 2\pi rH$ in the porous media.

Considering flow through a one-dimensional cylindrical medium of cross-sectional area A; from r to $r + \Delta r$ where Δr is a small change in the radius, r.

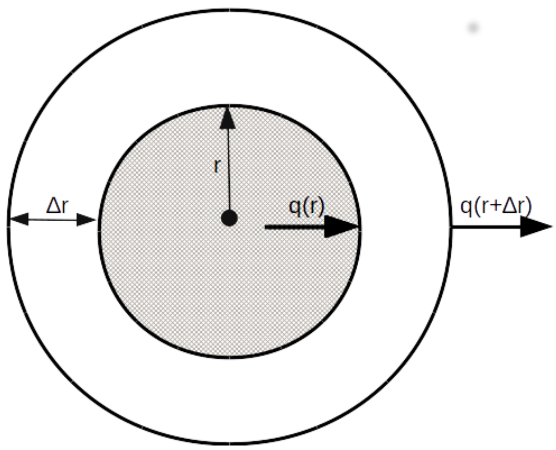


Figure 2.2: Annular flow pattern used to derive the diffusivity equation in radial coordinates

The main idea behind the application of the principle of conservation of mass is;

$$\textbf{Flux in - Flux out} = \textbf{Increase in amount stored} \qquad (2.8)$$

Considering the period of time between time t and $t+\Delta t$. Assuming that the fluid is flowing from left to right through the medium Δr. During the time increment (Δt), the mass flux into the region of porous rock at r can be expressed as:

$$\text{Mass flux in} = A_{(r)}\rho_{(r)}q_{(r)}\Delta t \tag{2.9}$$

The mass flux out of the porous rock at $(r + \Delta r)$ can be expressed as:

$$\text{Mass flux out} = A_{(r+\Delta r)}\rho_{(r+\Delta r)}q_{(r+\Delta r)}\Delta t \tag{2.10}$$

Let the amount of fluid mass stored in the region Δr be denoted by m, so the conservation of mass equation in time Δt takes the form

$$[2\pi r H\rho_{(r)}q_{(r)} - 2\pi(r + \Delta r)H\rho_{(r+\Delta r)}q_{(r+\Delta r)}]\Delta t = m_{(t+\Delta t)} - m_{(t)} \tag{2.11}$$

Dividing through Equation (2.11) by Δt and taking limits as $\Delta t \to 0$:

$$2\pi H[r\rho_{(r)}q_{(r)} - (r + \Delta r)\rho_{(r+\Delta r)}q_{(r+\Delta r)}]\Delta t = \lim_{\Delta t \to 0} \frac{m_{(t+\Delta t)} - m_{(t)}}{\Delta t} = \frac{\partial m}{\partial t} \tag{2.12}$$

But $m = \rho V_p$, where V_p is the pore volume of the rock contained in the region (Δr) between r and $r + \Delta r$.

So,

$$m = \rho V_p = \rho\phi V = \rho\phi 2\pi r H\Delta r$$

where; ϕ is the rock porosity and V is the rock bulk volume. Then;

$$\frac{\partial(\rho\phi 2\pi r H\Delta r)}{\partial t} = \frac{\partial(\rho\phi)}{\partial t}2\pi r H\Delta r$$

Therefore Equation (2.12) becomes;

$$2\pi H[r\rho_{(r)}q_{(r)} - (r + \Delta r)\rho_{(r+\Delta r)}q_{(r+\Delta r)}] = 2\pi r H\Delta r\frac{\partial(\rho\phi)}{\partial t} \tag{2.13}$$

Dividing through Equation (2.13) by $2\pi H\Delta r$, and taking limits as $\Delta r \to 0$ gives:

$$-\frac{\partial(r\rho q)}{\partial r} = r\frac{\partial(\rho\phi)}{\partial t} \tag{2.14}$$

Equation (2.14) is the basic equation of conservation of mass for 1-D radial flow in a porous medium. It is applicable to gases, liquids or high and low flow rate regimes. (Engler, 2010).

2.4.2 Darcy's law

Darcy's law presents a fundamental governing equation to describe the flow of fluids through porous media. The law was formulated by the French civil engineer, Henry Darcy in 1856 on the basis of his experiments on vertical water filtration through sand beds. Darcy found that his observations could be described mathematically by:

$$Q = \frac{CA\Delta(p - \rho gz)}{L}. \tag{2.15}$$

where: p = pressure ρ = density

g = gravitational acceleration z = vertical length from a reference datum

L = length of medium Q = volumetric flowrate

C = constant of proportionality A = cross-sectional area of medium.

Subsequent to Darcy's initial discovery, several authors including (Aziz and Settari, 1979) have found that, all other factors being equal, Q is inversely proportional to the fluid viscosity, μ. It is therefore convenient to factor out μ, and let $C = \frac{k}{\mu}$, where k is known as the absolute permeability of a single fluid through a porous medium. This has resulted to a more convenient way to express the law, with the volumetric flow per unit area given as, $q = Q/A$. Darcy's law is therefore usually written as:

$$q = \frac{Q}{A} = \frac{k}{\mu}\frac{\Delta(p - \rho gz)}{L} \tag{2.16}$$

where q is the fluid flux.(Aziz and Settari, 1979)

For transient processes in which the flux varies from one point to another, a differential form of Darcy's law is more appropriate. Assuming a horizontal flow of fluids, for example, the Darcy's law can be expressed as

$$q = \frac{Q}{A} = -\frac{k}{\mu}\frac{\partial(p - \rho g z)}{\partial x} \tag{2.17}$$

The minus(-) sign is conventional and accounts for the flow of fluids in opposite direction to pressure or potential gradient. Similarly, the Darcy's law for radial flow is expressed as:

$$q = \frac{Q}{A} = -\frac{k}{\mu}\frac{\partial(p - \rho g z)}{\partial r} \tag{2.18}$$

Darcy's law is the most widely used equation to describe the flow of fluid in porous media. However, a search through most fluid dynamics textbook suggests that, fluid motion is described by the Navier-Stokes equations. The key point to note is that, Darcy's law for porous media is derived from the Navier-Stokes equation under certain assumptions. By utilizing local averaging techniques-(Whitaker, 1986), or homogenization-(Hornung, 1997), it can be resolved that under appropriate assumptions the momentum conservation of the Navier-Stokes equation reduces to the Darcy's law on the macroscopic level.

2.5 Similarity Transformation

There exist several techniques for solving the PDEs associated with transient flow problems, such as the Fourier tansformation and Laplace transformation. However, the Boltzmann transformation is one which is widely used to analyze flow problems in reservoir engineering. This is a technique in which the dependence of a PDE on two or more independent variables is reduced to a particular combination of those variables such that the PDE is converted to an ODE, or to a

PDE with a smaller number of independent variables (Jiji, 2003). This process is also know as the change of variables or combination of variables method. One of such combination of variables is the Boltzmann variable, which is encountered in the solution of transient flow through porous media.

Similarity techniques is typically implemented after the single-phase diffusivity equation for a flow phenomenon has been linearized. This method can however be extended to multiphase flow problems nonetheless, since most oil reservoirs experience multiphase flow during production, and their flow equations are highly nonlinear (Tabatabaie and Pooladi-Darvish, 2016).

2.5.1 Boltzmann Variable

Boltzmann (1894) pioneered the use of similarity variables in his work on Fick's second law by converting it into an ordinary differential equation. Several classic engineering books utilize the concept of the similarity variable theory in order to establish analytical solutions of various phenomena of interest. Carslaw and Jaeger (1959), explored the use of the Boltzmann's transformation $\xi = r/\sqrt{t}$ to find solutions of heat conduction problems and employed several other variable transformations that made possible the establishment of analytical solutions. Bird et al. (2002) researched on several transport problems of fluid flow, heat and mass transfer. In their study, the use of the method of similarity approximations (also known as the method of combination of variables) was pivotal for the transformation of partial differential equations (PDEs) into one or more ordinary differential equations (ODEs). This approach was extended to attain analytical solutions.

The process of carrying out well test analysis in oil and gas engineering has been centered on the concept of the exponential integral solution, which yeilds upon applying the similarity theory to the partial differential equations; typically for single phase flow in porous media in a radial geometry. Over the years, the application of the similarity theory has proved beneficial in the area of reservoir engineering (Ayala and Kouassi, 2007).

2.6 Numerical Simulation of Fluid Flow in Hydrocarbon Reservoirs

Attempts to finding exact solutions to some of the differential equations describing fluid flow in porous media prove somewhat impossible or very time consuming to carry out (Kamyabi, 2014). Therefore numerical analysis provides a means to address this issue by creating a sequence of approximants to the exact solution to solve the flow problem. Numerical approaches are initiated through discretization of the continuous differential equations. Although there exist several methods that offer means to obtain these approximate solutions to flow problems, it appears that the most widely used method is the Finite Difference Method (FDM) which is elaborated in detail by Aziz and Settari (1979).

2.6.1 Discretization

In the finite difference method, the derivatives in differential equations are approximated using the Taylor's expansion. Forward, backward and central differences are the three forms in this method commonly used to obtain the approximations of numerical derivatives. Thus, derivatives spanning from differential equations are substituted with finite difference equations (Stevenson et al., 1991).

Considering a function w expressed in terms of x. Given the nodal values of as shown in Figure (2.3), the first-order derivatives of w with respect x can be approximated by:

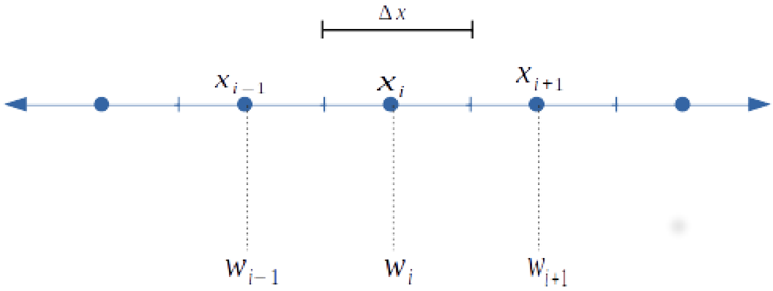


Figure 2.3: Illustration of grid point discretization in 1-Dimension

$$\frac{\partial w}{\partial x} = \frac{w_{i+1} - w_i}{\Delta x} + O(\Delta x) \tag{2.19}$$

Or

$$\frac{\partial w}{\partial x} = \frac{w_i - w_{i-1}}{\Delta x} + O(\Delta x) \tag{2.20}$$

Equation (2.19) and Equation (2.20) represent the forward difference and backward difference approximations respectively of the first order derivative of w in terms of x. These two approximations are collectively called first-order approximations since their remainders (truncation error) are in order of $O(\Delta x)$- First order. However, taking the average of the forward and backward differences yields the central difference approximation, which has an order of $O(\Delta x)^2$-Second order. Hence, this makes the central difference more precise than the previous approximations (Kamyabi, 2014). The central difference approximation for the first order derivative of w in terms of x is given as:

$$\frac{\partial w}{\partial x} = \frac{w_{i+1} - w_{i-1}}{2\Delta x} + O(\Delta x)^2 \tag{2.21}$$

Similarly, the second order derivative of w in terms of x can be expressed as:

$$\frac{\partial w}{\partial x} = \frac{w_{i+1} - 2w_i + w_{i-1}}{(\Delta x)^2} + O(\Delta x)^2 \tag{2.22}$$

Several finite difference formulations are in existence to approximate higher order derivatives as well as multi-dimensional derivatives using the combination of one dimensional finite difference approximations in different dimensions.(Aziz and Settari, 1979)

1. **Spatial discretization** involves the division of the continuous simulation domain into grid points or blocks depending on the order of the spatial dimension (Kamyabi, 2014). The interval(size) between grid points(blocks) is important in the running time and consistency of fluid flow simulations. Depending on the system complexity, the interval lengths or block sizes can be uniform or variable (Stevenson et al., 1991). In a more general sense, fine spatial discretizations typically yield better approximations compared to coarse spatial discretizations.

2. **Temporal discretization** is carried out by dividing simulation time into timesteps. It is suitable that, timestep can neither be too short because of the computation restrictions nor too big due to the consistency issues (Kamyabi, 2014). From the viewpoint of reservoir engineering time discretization schemes must be stable, robust and computationally efficient. Forward differencing (explicit) methods, for example, are only stable for time steps constrained by:

$$\Delta t \leq k(\Delta x)^2$$

   where, Δt and Δx are the temporal and spatial intervals respectively and k is some constant. Implicit methods on the other hand are unconditionally stable.(Stevenson et al., 1991)

2.6.2 Courant-Friedrichs-Lewy condition

The Courant-Friedrichs-Lewy (CFL) condition is an essential condition that institutes convergence when numerically solving certain partial differential equations (PDEs) by the method of finite differences. This condition is essential in the numerical analysis of explicit time schemes. The condition is based on the Courant number which is a dimensionless number expressed as a function of timestep, gridblock size or grid-point interval, and the velocity at that gridblock or point (Trefethen, 1994). The Courant number is expressed as:

$$c = \Delta t \sum_{i=1}^{n} \frac{u_{x_i}}{\Delta x_i} \leq C_{max} \tag{2.23}$$

where the indices i and n show the current and maximum values of dimension in space. This equation implies that the solution is more stable at small values of Courant number as well as timestep on the condition that, the method used is not unconditionally stable.

The value of C_{max} changes with the method used to solve the discretized equation, especially depending on whether the method is explicit or implicit. If an explicit scheme is used then typically $C_{max} = 1$. Implicit schemes are usually less sensitive to numerical instability and so larger values of C_{max} may be satisfactory (Kamyabi, 2014).

2.6.3 Linearization Schemes

The discretization of the differential equations describing multiphase flow in hydrocarbon reservoirs produces non-linear coupled difference equations which need to be properly resolved. Some of the linearization techniques utilized to handle non-linearities in reservoir simulation include; fully Implicit (FI), Implicit-Pressure-Explicit-Saturation (IMPES), and Adaptive Implicit Methods (AIM). In this work, the IMPES method is adopted.

Implicit Pressure Explicit Saturation (IMPES)

Sheldon B. Aker (1959) is among the first researchers to implement the IMPES method in their works. This method yields less computational error and it is computationally fast when implemented making it more favorable (Kamyabi, 2014). The underlining principle of this technique applied in multiphase flow is to decouple the problem into the pressure and saturation equations after a combination of the flow equations. After the pressure has been advanced in time, the fluid saturations are updated explicitly. This process is repeated for the entire simulation time allowable (Peaceman, 1977).

Chapter 3

Methodology

Among the techniques usually used to solve transient fluid flow problems, the similarity solution is one which is widely used in the petroleum engineering literature. In this chapter, the study presents the model that describes the multiphase flow in tight oil reservoirs under some assumptions.

In subsequent sections, it is shown that the simultaneous flow of oil and gas in one dimensional cylindrical coordinates can be converted to two nonlinear ODEs for pressure and saturation when the Boltzmann variable is used. This means that the pressure and saturation solutions can be written as unique functions of the Boltzmann variable, requiring that saturation be considered as solely a function of pressure.

3.1 Radial Diffusivity Equation

The radial diffusivity equation is considered one of the most salient and widely used mathematical expressions in the petroleum industry. The equation is particularly applied to the analysis of well testing data. In this study, the radial diffusivity equation is used as the basic block for constructing the flow model. Under a set of assumptions placed on the flow phenomenon, the three (3) governing equations needed in deriving the radial diffusivity equation are combined. These governing equations include:

1. The law of mass conservation (Continuity equation).
2. Darcy's empirical law.
3. Equation of state.

3.1.1 Assumptions

The mathematical model considered in this study describes the isothermal radial flow of oil and gas under the following assumptions:

1. Formation is a homogeneous and isotropic porous media of uniform thickness.
2. Reservior volume drained by the well is circular ,horizontal and of constant thickness
3. A central well is perforated across the entire formation thickness. Hence; radial flow.
4. No oil is dissolved in the gas phase
5. Capillary and gravity effects are negligible.
6. Reservoir producing at a constant flowing pressure.
7. Water saturation is assumed to be immobile and is considered to be a part of the rock volume.
8. Flow region is free of sources and sinks.

3.2 Mathematical Formulation

Here, the flow of oil and gas in porous media is described by the well known Black Oil formulation applied to two-phase fluid flow (Aziz and Settari, 1979). A combination of the mass conservation equations (continuity equation) with Darcy's empirical law yield differential equations that describe the flow of hydrocarbons in a reservoir. In this study, the equations used in describing the 1-dimensional radial flow of oil and gas respectively, are represented by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{kk_{ro}}{\mu_o B_o}\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right) \tag{3.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rk\left(R_s\frac{k_{ro}}{\mu_o B_o} + \frac{k_{rg}}{\mu_g B_g}\right)\frac{\partial p}{\partial r}\right] = \frac{\partial}{\partial t}\left[\phi\left(\frac{R_s S_o}{B_o} + \frac{S_g}{B_g}\right)\right] \tag{3.2}$$

where:

r=radius k=absolute permeability

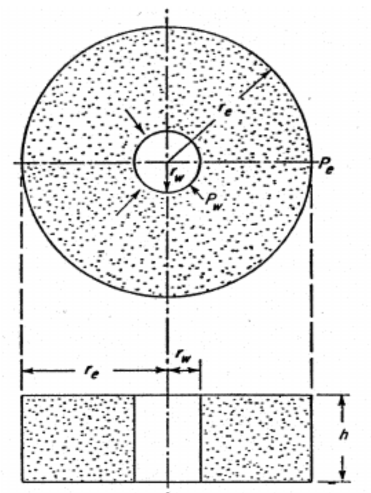
ϕ=porosity B_i=phase formation volume factor

μ_i=phase viscosity S_i=phase saturation

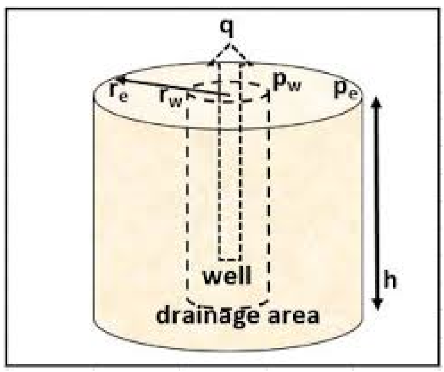
R_s= Solution gas ratio k_{ri}=phase relative permeability

i= indicator for fluid phases(oil(o) and gas(g))

The fluid flow, Equation (3.1) and Equation (3.2) present a set of coupled, nonlinear PDEs. The basic equations can be mathematically manipulated into several alternate forms with various choices of primary dependent variables (Chen and Ewing, 1997). The choice of equation form and primary solution defined by variables have considerable implications for the mathematical analysis and the numerical method used to solve these equations (Chen and Ewing, 1997).



(a) A 1D flow geometry



(b) A 3D flow geometry

Figure 3.1: Radial flow representation of the hydrocarbons into the wellbore.(NomadicGeo, 2016)

Equation (3.1) and Equation (3.2) represent radial flow of oil and gas respectively. A pictorial perspective of the flow geometry is shown in Figure (3.1). Considering the equation describing gas flow, its worthwhile to note how the PDE accounts for both gases dissolved in solution and those that evolve out of the solution when in-situ pressures fall below bubble point pressure. This is accomplished by utilizing the solution oil-gas ratio (R_s). All other parameters and symbols are defined in the nomenclature.

3.3 Similarity Transformation of The Model

In this section, a step-wise procedure is performed to facilitate the similarity transformation of the diffusivity equations into ODEs using the Boltzmann variable. For the purpose of a quick transformation, the original equations (Equation (3.1) and Equation (3.2)) are reduced to simple forms using the parameters defined in Table (3.1).

Table 3.1: Definition of Parameters

Parameter	Definition
α	$\dfrac{kk_{ro}}{\mu_o B_o}$
β	$\dfrac{\phi S_o}{B_o}$
b	$\dfrac{R_s S_o}{B_o} + \dfrac{S_g}{B_g}$
R	$R_s + \dfrac{k_{rg}\mu_o B_o}{k_{ro}\mu_g B_g}$

3.3.1 Similarity Transformation of the Oil Equation

First, the diffusivity equation describing one-dimensional radial flow of oil, which is given as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{kk_{ro}}{\mu_o B_o}\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right) \tag{3.3}$$

is simplified to Equation (3.6) by substituting the parameters given by Equation (3.4) and Equation (3.5)

$$\alpha = \frac{kk_{ro}}{\mu_o B_o} \tag{3.4}$$

$$\beta = \frac{\phi S_o}{B_o} \tag{3.5}$$

The reduced form of Equation (3.3) to Equation (3.6) presents the equation upon which the similarity transformation is carried out.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha\frac{\partial p}{\partial r}\right) = \frac{\partial \beta}{\partial t} \tag{3.6}$$

At this point, the Boltzmann variable (given as $\xi = \frac{r}{\sqrt{t}}$) is introduce. In order to carry out the transformation, the first derivatives of ξ with respect to r and t are required and are given as:

$$\frac{\partial \xi}{\partial r} = \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{t}} = \frac{1}{r}\frac{r}{\sqrt{t}} = \frac{\xi}{r} \tag{3.7}$$

$$\frac{\partial^2 \xi}{\partial r^2} = 0 \tag{3.8}$$

$$\frac{\partial \xi}{\partial t} = -\frac{r}{\sqrt{t}}\frac{1}{2t} = -\frac{\xi}{2t} \tag{3.9}$$

From Equation (3.6), expanding the left hand side yields:

$$\frac{\partial}{\partial r}\left(\alpha\frac{\partial p}{\partial r}\right) + \frac{1}{r}\left(\alpha\frac{\partial p}{\partial r}\right) = \frac{\partial \beta}{\partial t} \tag{3.10}$$

A change of variables is performed by implementing the chain rule on Equation (3.10),to introduce the derivatives of the Boltzmann variable.

$$\frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi}\left(\alpha\frac{\partial \xi}{\partial r}\frac{\partial p}{\partial \xi}\right) + \frac{1}{r}\left(\alpha\frac{\partial \xi}{\partial r}\frac{\partial p}{\partial \xi}\right) = \frac{\partial \xi}{\partial t}\frac{\partial \beta}{\partial \xi} \tag{3.11}$$

Further expansion of Equation (3.11) gives:

$$\frac{\partial \xi}{\partial r}\left[\frac{\partial}{\partial \xi}\left(\alpha\frac{\partial \xi}{\partial r}\right)\frac{\partial p}{\partial \xi}+\alpha\frac{\partial \xi}{\partial r}\frac{\partial^2 p}{\partial \xi^2}\right]+\frac{\alpha}{r}\frac{\partial \xi}{\partial r}\frac{\partial p}{\partial \xi}=\frac{\partial \xi}{\partial t}\frac{\partial \beta}{\partial \xi} \tag{3.12}$$

By grouping terms, Equation (3.12) gives:

$$\alpha\left(\frac{\partial \xi}{\partial r}\right)^2\frac{\partial^2 p}{\partial \xi^2}+\left[\frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi}\left(\alpha\frac{\partial \xi}{\partial r}\right)+\frac{\alpha}{r}\frac{\partial \xi}{\partial r}\right]\frac{\partial p}{\partial \xi}=\frac{\partial \xi}{\partial t}\frac{\partial \beta}{\partial \xi} \tag{3.13}$$

436 Dividing through Equation (3.12) by $\left(\frac{\partial \xi}{\partial r}\right)^2$ and expanding derivatives gives:

$$\alpha\frac{\partial^2 p}{\partial \xi^2}+\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2}\left[\frac{\partial}{\partial r}\left(\alpha\frac{\partial \xi}{\partial r}\right)+\frac{\alpha}{r}\frac{\partial \xi}{\partial r}\right]\frac{\partial p}{\partial \xi}=\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2}\frac{\partial \xi}{\partial t}\frac{\partial \beta}{\partial \xi} \tag{3.14}$$

$$\alpha\frac{\partial^2 p}{\partial \xi^2}+\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2}\left[\frac{\partial \alpha}{\partial r}\frac{\partial \xi}{\partial r}+\alpha\frac{\partial^2 \xi}{\partial r^2}+\frac{\alpha}{r}\frac{\partial \xi}{\partial r}\right]\frac{\partial p}{\partial \xi}=\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2}\frac{\partial \xi}{\partial t}\frac{\partial \beta}{\partial \xi} \tag{3.15}$$

$$\alpha\frac{\partial^2 p}{\partial \xi^2}+\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2}\left[\frac{\partial \xi}{\partial r}\frac{\partial \alpha}{\partial xi}\frac{\partial \xi}{\partial r}+\alpha\frac{\partial^2 \xi}{\partial r^2}+\frac{\alpha}{r}\frac{\partial \xi}{\partial r}\right]\frac{\partial p}{\partial \xi}=\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2}\frac{\partial \xi}{\partial t}\frac{\partial \beta}{\partial \xi} \tag{3.16}$$

From Equation (3.8), $\frac{\partial^2 \xi}{\partial r^2}=0$,hence Equation (3.16) simplifies to:

$$\alpha\frac{\partial^2 p}{\partial \xi^2}+\left[\frac{\partial \alpha}{\partial \xi}+\frac{\alpha}{r}\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)}\right]\frac{\partial p}{\partial \xi}=\frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2}\frac{\partial \xi}{\partial t}\frac{\partial \beta}{\partial \xi} \tag{3.17}$$

437 Substituting Equation (3.7) and Equation (3.9) into Equation (3.17) gives:

$$\alpha\frac{\partial^2 p}{\partial \xi^2}+\left[\frac{\partial \alpha}{\partial \xi}+\frac{\alpha}{\xi}\right]\frac{\partial p}{\partial \xi}=-\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} \tag{3.18}$$

$$\alpha\frac{\partial^2 p}{\partial \xi^2}+\frac{\partial \alpha}{\partial \xi}\frac{\partial p}{\partial \xi}+\frac{\alpha}{\xi}\frac{\partial p}{\partial \xi}=-\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} \tag{3.19}$$

The first and second terms of Equation 3.19 are re-written to yield an equation (Equation 3.20) for the oil equation in terms of the Boltzmann variable. This concludes the similarity transformation of the oil equation.

$$\frac{\partial}{\partial \xi}\left(\alpha \frac{\partial p}{\partial \xi}\right) + \frac{\alpha}{\xi}\frac{\partial p}{\partial \xi} = -\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} \tag{3.20}$$

Since the PDE Equation (3.20) is only dependent on the Boltzmann variable it is re-written as an ODE in the form:

$$\frac{d}{d\xi}\left(\alpha \frac{dp}{d\xi}\right) + \frac{\alpha}{\xi}\frac{dp}{d\xi} = -\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} \tag{3.21}$$

3.3.2 Similarity Transformation of the Gas Equation

Next, the diffusivity equation describing one-dimensional radial flow of gas is considered.

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rk\left(R_s\frac{k_{ro}}{\mu_o B_o} + \frac{k_{rg}}{\mu_g B_g}\right)\frac{\partial p}{\partial r}\right] = \frac{\partial}{\partial t}\left[\phi\left(\frac{R_s S_o}{B_o} + \frac{S_g}{B_g}\right)\right] \tag{3.22}$$

By using the following parameters, the diffusivity equation given by Equation 3.22) is simplified to Equation 3.26)

$$\alpha = \frac{kk_{ro}}{\mu_o B_o} \tag{3.23}$$

$$b = \frac{R_s S_o}{B_o} + \frac{S_g}{B_g} \tag{3.24}$$

$$R = R_s + \frac{k_{rg}\mu_o B_o}{k_{ro}\mu_g B_g} \tag{3.25}$$

The reduced equation is obtained as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rR\alpha\frac{\partial p}{\partial r}\right) = \frac{\partial b}{\partial t} \tag{3.26}$$

Again, the Boltzmann variable (given as $\xi = \frac{r}{\sqrt{t}}$) is introduce to carry out the similarity transformation. The first derivatives of ξ with respect to r and t necessary to perform this task which are obtained below.

$$\frac{\partial \xi}{\partial r} = \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{t}} = \frac{1}{r}\frac{r}{\sqrt{t}} = \frac{\xi}{r} \tag{3.27}$$

$$\frac{\partial^2 \xi}{\partial r^2} = 0 \tag{3.28}$$

$$\frac{\partial \xi}{\partial t} = -\frac{r}{\sqrt{t}}\frac{1}{2t} = -\frac{\xi}{2t} \tag{3.29}$$

Now, expanding the left hand side of Equation 3.26) yields:

$$\frac{\partial}{\partial r}\left(R\alpha\frac{\partial p}{\partial r}\right) + \frac{1}{r}\left(R\alpha\frac{\partial p}{\partial r}\right) = \frac{\partial b}{\partial t} \tag{3.30}$$

By implementing the chain rule on Equation (3.30), a change of variable is brought forth to enable the introduction of the derivatives of the Boltzmann variable.

$$\frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi}\left(R\alpha\frac{\partial \xi}{\partial r}\frac{\partial p}{\partial \xi}\right) + \frac{1}{r}\left(R\alpha\frac{\partial \xi}{\partial r}\frac{\partial p}{\partial \xi}\right) = \frac{\partial \xi}{\partial t}\frac{\partial b}{\partial \xi} \tag{3.31}$$

Further expansion of Equation (3.31) gives:

$$\frac{\partial \xi}{\partial r}\left[\frac{\partial}{\partial \xi}\left(R\alpha\frac{\partial \xi}{\partial r}\right)\frac{\partial p}{\partial \xi} + R\alpha\frac{\partial \xi}{\partial r}\frac{\partial^2 p}{\partial \xi^2}\right] + \frac{R\alpha}{r}\frac{\partial \xi}{\partial r}\frac{\partial p}{\partial \xi} = \frac{\partial \xi}{\partial t}\frac{\partial b}{\partial \xi} \tag{3.32}$$

By grouping terms, Equation (3.32) gives:

$$R\alpha\left(\frac{\partial \xi}{\partial r}\right)^2\frac{\partial^2 p}{\partial \xi^2} + \left[\frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi}\left(R\alpha\frac{\partial \xi}{\partial r}\right) + \frac{R\alpha}{r}\frac{\partial \xi}{\partial r}\right]\frac{\partial p}{\partial \xi} = \frac{\partial \xi}{\partial t}\frac{\partial b}{\partial \xi} \tag{3.33}$$

Dividing through Equation (3.33) by $\left(\frac{\partial \xi}{\partial r}\right)^2$ and expanding derivatives gives:

$$R\alpha \frac{\partial^2 p}{\partial \xi^2} + \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2} \left[\frac{\partial}{\partial r} \left(R\alpha \frac{\partial \xi}{\partial r} \right) + \frac{R\alpha}{r} \frac{\partial \xi}{\partial r} \right] \frac{\partial p}{\partial \xi} = \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2} \frac{\partial \xi}{\partial t} \frac{\partial \beta}{\partial \xi} \tag{3.34}$$

$$R\alpha \frac{\partial^2 p}{\partial \xi^2} + \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2} \left[\frac{\partial (R\alpha)}{\partial r} \frac{\partial \xi}{\partial r} + R\alpha \frac{\partial^2 \xi}{\partial r^2} + \frac{R\alpha}{r} \frac{\partial \xi}{\partial r} \right] \frac{\partial p}{\partial \xi} = \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2} \frac{\partial \xi}{\partial t} \frac{\partial \beta}{\partial \xi} \tag{3.35}$$

$$R\alpha \frac{\partial^2 p}{\partial \xi^2} + \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2} \left[\frac{\partial \xi}{\partial r} \frac{\partial (R\alpha)}{\partial xi} \frac{\partial \xi}{\partial r} + R\alpha \frac{\partial^2 \xi}{\partial r^2} + \frac{R\alpha}{r} \frac{\partial \xi}{\partial r} \right] \frac{\partial p}{\partial \xi} = \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2} \frac{\partial \xi}{\partial t} \frac{\partial \beta}{\partial \xi} \tag{3.36}$$

From Equation (3.28), $\frac{\partial^2 \xi}{\partial r^2} = 0$, hence Equation (3.36) simplifies to:

$$R\alpha \frac{\partial^2 p}{\partial \xi^2} + \left[\frac{\partial (R\alpha)}{\partial \xi} + \frac{R\alpha}{r} \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)} \right] \frac{\partial p}{\partial \xi} = \frac{1}{\left(\frac{\partial \xi}{\partial r}\right)^2} \frac{\partial \xi}{\partial t} \frac{\partial b}{\partial \xi} \tag{3.37}$$

Substituting Equation (3.27) and Equation (3.29) into Equation (3.37) gives:

$$R\alpha \frac{\partial^2 p}{\partial \xi^2} + \left[\frac{\partial (R\alpha)}{\partial \xi} + \frac{R\alpha}{\xi} \right] \frac{\partial p}{\partial \xi} = -\frac{\xi}{2} \frac{\partial b}{\partial \xi} \tag{3.38}$$

$$R\alpha \frac{\partial^2 p}{\partial \xi^2} + \frac{\partial (R\alpha)}{\partial \xi} \frac{\partial p}{\partial \xi} + \frac{R\alpha}{\xi} \frac{\partial p}{\partial \xi} = -\frac{\xi}{2} \frac{\partial \beta}{\partial \xi} \tag{3.39}$$

Expanding the derivative of the second term in Equation (3.39) gives:

$$R\alpha \frac{\partial^2 p}{\partial \xi^2} + R \frac{\partial \alpha}{\partial \xi} \frac{\partial p}{\partial \xi} + \alpha \frac{\partial R}{\partial \xi} \frac{\partial p}{\partial \xi} + \frac{R\alpha}{\xi} \frac{\partial p}{\partial \xi} = -\frac{\xi}{2} \frac{\partial b}{\partial \xi} \tag{3.40}$$

Equation (3.40) is re-written as:

$$\alpha \frac{\partial R}{\partial \xi} \frac{\partial p}{\partial \xi} + R \frac{\partial}{\partial \xi} \left(\alpha \frac{\partial p}{\partial \xi} \right) + \frac{R\alpha}{\xi} \frac{\partial p}{\partial \xi} = -\frac{\xi}{2} \frac{\partial b}{\partial \xi} \tag{3.41}$$

Recalling the oil equation(Equation (3.20)) given below

$$\frac{\partial}{\partial \xi}\left(\alpha \frac{\partial p}{\partial \xi}\right) + \frac{\alpha}{\xi}\frac{\partial p}{\partial \xi} = -\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} \tag{3.42}$$

and further expanded to:

$$\Longrightarrow \frac{\partial}{\partial \xi}\left(\alpha \frac{\partial p}{\partial \xi}\right) = -\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} - \frac{\alpha}{\xi}\frac{\partial p}{\partial \xi} \tag{3.43}$$

Substituting Equation (3.43) into Equation (3.41) gives:

$$\alpha \frac{\partial R}{\partial \xi}\frac{\partial p}{\partial \xi} + R\left[-\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} - \frac{\alpha}{\xi}\frac{\partial p}{\partial \xi}\right] + \frac{R\alpha}{\xi}\frac{\partial p}{\partial \xi} = -\frac{\xi}{2}\frac{\partial b}{\partial \xi} \tag{3.44}$$

Expanding and simplifying terms in Equation (3.44) yields a diffusivity equation in terms of the Boltzmann variable. This is the similarity transform of the gas equation

$$\alpha \frac{\partial R}{\partial \xi}\frac{\partial p}{\partial \xi} = \frac{\xi}{2}\left[R\frac{\partial \beta}{\partial \xi} - \frac{\partial b}{\partial \xi}\right] \tag{3.45}$$

Since the PDE Equation (3.45) is only dependent on the Boltzmann variable,it is re-written as an ODE in the form:

$$\alpha \frac{\partial R}{\partial \xi}\frac{dp}{d\xi} = \frac{\xi}{2}\left[R\frac{\partial \beta}{\partial \xi} - \frac{\partial b}{\partial \xi}\right] \tag{3.46}$$

3.3.3 Summary

The Boltzmann variable which is given as $\xi = \dfrac{r}{\sqrt{t}}$ is used to carry out the similarity transformation of the flow problem. The Boltzmann variable is introduced to convert Equation (3.1) and Equation (3.2) into a coupled pair of ODEs:

$$\frac{d}{d\xi}\left(\alpha \frac{dp}{d\xi}\right) + \frac{\alpha}{\xi}\frac{dp}{d\xi} = -\frac{\xi}{2}\frac{\partial \beta}{\partial \xi} \tag{3.47}$$

$$\alpha \frac{\partial R}{\partial \xi}\frac{dp}{d\xi} = \frac{\xi}{2}\left[R\frac{\partial \beta}{\partial \xi} - \frac{\partial b}{\partial \xi}\right] \tag{3.48}$$

From Equation (3.1) through Equation (3.48), p and S_o represent the pressure and oil saturation, respectively, and all other parameters (α, β, b, and R) are functions of pressure and saturation, as defined in Table (3.1). Note that all partial derivatives in Equation (3.47) and Equation (3.48) are related to Pressure-Volume-Temperature (PVT) properties and relative permeability functions; which are all assumed to be known.

Equation (3.47) and Equation (3.48) are then be solved numerically as a system of two equations and two unknowns to find pressure and saturation profiles under suitable boundary conditions. This process yields the semi-analytic similarity solution of the problem.

3.4 Semi-Analytic Similarity Solution

In this section, the semi-analytical solutions to the ODEs derived from the similarity transformation is developed for pressure and saturation. First, a pressure and saturation equations are obtained

3.4.1 Pressure Equation

From Equation (3.47), the ODE for oil flow is expressed as:

$$\frac{d}{d\xi}\left(\alpha\frac{dp}{d\xi}\right)+\frac{\alpha}{\xi}\frac{dp}{d\xi}=-\frac{\xi}{2}\frac{\partial\beta}{\partial\xi} \tag{3.49}$$

Since β is a function of p and So, Equation (3.49) expanded to:

$$\frac{d}{d\xi}\left(\alpha\frac{dp}{d\xi}\right)+\frac{\alpha}{\xi}\frac{dp}{d\xi}=-\frac{\xi}{2}\left[\frac{\partial\beta}{\partial p}\frac{dp}{d\xi}+\frac{\partial\beta}{\partial So}\frac{dSo}{d\xi}\right] \tag{3.50}$$

Equation (3.50) has two unknowns, pressure and saturation. If saturation were known, Equation (3.50) could be solved directly to find pressure as a function of the Boltzmann variable. However, saturation profile is not known at priori. In order to complete the system of equations and unknowns, Equation (3.48) of gas is employed to obtain a saturation equation.

3.4.2 Saturation Equation

From Equation (3.48), the ODE for gas flow is expressed as:

$$\alpha \frac{\partial R}{\partial \xi}\frac{dp}{d\xi} = \frac{\xi}{2}\left[R\frac{\partial \beta}{\partial \xi} - \frac{\partial b}{\partial \xi}\right] \tag{3.51}$$

Since R, b and β are a function of p and So, Equation (3.51) is expanded to:

$$\alpha\left(\frac{\partial R}{\partial p}\frac{dp}{d\xi} + \frac{\partial R}{\partial So}\frac{So}{d\xi}\right)\frac{dp}{d\xi} = \frac{\xi}{2}\left[R\left(\frac{\partial \beta}{\partial p}\frac{dp}{d\xi} + \frac{\partial \beta}{\partial So}\frac{So}{d\xi}\right) - \left(\frac{\partial b}{\partial p}\frac{dp}{d\xi} + \frac{\partial b}{\partial So}\frac{So}{d\xi}\right)\right] \tag{3.52}$$

Re-arranging the Equation (3.51) yields the saturation equation as:

$$\frac{dS_o}{d\xi} = -\frac{dp}{d\xi}\left[\frac{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial p} - \xi\left(R\frac{\partial \beta}{\partial p} - \frac{\partial b}{\partial p}\right)}{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial S_o} - \xi\left(R\frac{\partial \beta}{\partial S_o} - \frac{\partial b}{\partial S_o}\right)}\right] \tag{3.53}$$

3.4.3 Boundary Conditions

One initial condition on saturation and two boundary conditions on pressure are sufficient to solve Equation (3.50) and Equation (3.53) simultaneously. Equation (3.50) and Equation (3.53) constitute a system of two equations and two unknowns which can be solved together to find pressure and saturation profiles as a function of the Boltzmann variable. it can be observed from Equation (3.50) and Equation (3.53) that, the pressure and saturation are unique functions of the Boltzmann variable, ξ.

3.4.4 Solving for p and S_o in terms of ξ

In order to numerically solve the ODEs of Equation (3.50) and Equation (3.53), the non linearity introduced by α and R are assumed negligible. This simplifies the ODEs to:

$$\alpha \frac{d^2p}{d\xi^2} + \frac{\alpha}{\xi}\frac{dp}{d\xi} = -\frac{\xi}{2}\left(\frac{\partial \beta}{\partial p}\frac{dp}{d\xi} + \frac{\partial \beta}{\partial S_o}\frac{dS_o}{d\xi}\right) \tag{3.54}$$

$$\frac{dS_o}{d\xi} = -\frac{dp}{d\xi}\left[\frac{\left(R\dfrac{\partial \beta}{\partial p} - \dfrac{\partial b}{\partial p}\right)}{\left(R\dfrac{\partial \beta}{\partial S_o} - \dfrac{\partial b}{\partial S_o}\right)}\right] \tag{3.55}$$

The resulting ODEs (Equation (3.54) and Equation (3.55)) facilitate the determination of pressure and saturation profiles exiting in the reservoir. This procedure offers a quick calculation of the reservoir performance and avoids the lengthy calculations inherent in solving the original PDEs. Equation (3.54) and Equation (3.55) are solved numerically by the finite difference method. The steps adopted in executing this task are as follows:

1. Substitute Equation (3.55) into Equation (3.54)

2. Discretize the resulting equation and Equation (3.55) using the finite difference approximations for the derivatives and assign uniform saturation and pressure profiles, equal to the initial saturation and pressure, to all grid points.

3. Solve the equation found in Step 1 together with the appropriate boundary conditions to find the pressure profile.

4. Using the pressure profile found in the previous step; solve Equation (3.55) to find the saturation profile.

5. Using the new saturation profile found in the previous step, calculate the pressure profile from the equation obtained in Step 1.

6. Steps 4 and 5 are repeated until convergence is achieved.

3.5 Full Numerical Solution

The PDEs associated with the flow phenomenon in this work are highly non-linear. Due to this challenge, the solution to the PDEs are obtained by a numerical approximation. This section briefly accounts for the full numerical approach undertaken to solve the PDEs describing the flow of oil and gas. Here, the differential equations are solved, by adopting the IMPES (Implicit-Pressure-Explicit-Saturation) scheme.

The fundamental principle of the IMPES method is to obtain a single pressure equation by a combination of the flow equations. Once pressure is implicitly computed for the new time, saturation is updated explicitly. The following is a brief description of the initial steps undertaken to solve the PDE system given by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{kk_{ro}}{\mu_o B_o}\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right) \tag{3.56}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rk\left(R_s\frac{k_{ro}}{\mu_o B_o} + \frac{k_{rg}}{\mu_g B_g}\right)\frac{\partial p}{\partial r}\right] = \frac{\partial}{\partial t}\left[\phi\left(\frac{R_s S_o}{B_o} + \frac{S_g}{B_g}\right)\right] \tag{3.57}$$

Similar to the earlier approach, Equation (3.56) and Equation (3.57) are used to derive pressure and saturation equations, which are necessary to solve for the two unknowns (pressure and saturation) under suitable boundary conditions.

3.5.1 Pressure Equation

The first step is to obtain the pressure equation, by combining flow equations of the oil and gas as follows:
Equation (3.56) multiplied by $(B_w - R_s B_g)$ and Equation (3.57) multiplied by Bg are added. In this way, the right hand side (RHS) of the resulting equation

is:

$$B_g \frac{\partial}{\partial t}\left[\phi\left(R_s \frac{S_o}{B_o} + \frac{S_g}{B_g}\right)\right] + (B_o - RsB_g)\frac{\partial}{\partial t}\left[\phi \frac{S_o}{B_o}\right] \tag{3.58}$$

Using the chain rule to expand the time derivatives of the obtained expression and carrying out some computations and rearrangements, further simplifies the right hand side expression to:

$$\phi\left[S_g\left(-\frac{1}{B_g}\frac{\partial B_g}{\partial p}\right) + S_g\left(-\frac{1}{B_o}\frac{\partial B_o}{\partial p} + \frac{B_g}{B_o}\frac{\partial R_s}{\partial p}\right)\right]\frac{\partial p}{\partial t} \tag{3.59}$$

Here, in Equation (3.58) all time derivatives of saturation resolve out. This is because the state of the reservoir requires that; $S_o + S_g = 1$

Taking note of the following mathematically derived approximate compressibilities, that is:

1. Gas compressibility: $c_g = -\dfrac{1}{B_g}\dfrac{\partial B_g}{\partial p}$
2. Oil compressibility: $c_o = -\dfrac{1}{B_o}\dfrac{\partial B_o}{\partial p} + \dfrac{B_g}{B_o}\dfrac{\partial R_s}{\partial p}$
3. Total compressibility: $c_t = S_g c_g + S_o c_o$

the right hand side is expressed as:

$$\phi c_t \frac{\partial p}{\partial t} \tag{3.60}$$

where, the formation compressibility c_f is considered negligible.

Introducing the left hand side (LHS) of the combined Equations ((3.56) and (3.57)) gives the pressure equation as:

$$(B_o - RsB_g)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{kk_{ro}}{\mu_o B_o}\frac{\partial p}{\partial r}\right) + B_g\frac{1}{r}\frac{\partial}{\partial r}\left[rk\left(R_s\frac{k_{ro}}{\mu_o B_o} + \frac{k_{rg}}{\mu_g B_g}\right)\frac{\partial p}{\partial r}\right] = \phi c_t \frac{\partial p}{\partial t} \tag{3.61}$$

Equation (3.61) is further reduced by introducing the parameters given in Table (3.1) to:

$$(B_o - RsB_g)\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha\frac{\partial p}{\partial r}\right) + B_g\frac{1}{r}\frac{\partial}{\partial r}\left(rR\alpha\frac{\partial p}{\partial r}\right) = \phi c_t\frac{\partial p}{\partial t} \tag{3.62}$$

3.5.2 Saturation Equation

Equation (3.56) which describe the radial flow of oil given below as;

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{kk_{ro}}{\mu_o B_o}\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right) \tag{3.63}$$

is used to determine the saturation profile once the pressure profile is obtained from Equation (3.62)

3.5.3 Solving for p and S_o in terms of r and t

By the finite difference method, Equation (3.62)) and Equation (3.63) are discretized. The system of equations that result is linearized by evaluating the pressure and saturation dependent functions (Pressure-Volume-Temperature parameters, fluid viscosities and relative permeabilities) in the pressure and saturation values of the previous time step.

The pressure equation is solved implicitly while the saturation equation is solved explicitly (IMPES scheme). This is carried out under a suitable stability restrictions for appropriate time steps (Maciasa et al., 2013). To accomplish this task, a computer algorithm is is written to carry out the computation of the system of equations that evolve from the problem.

3.5.4 Flowchart of the Algorithm for the Numerical Approach

The computational tool adopts the IMPES method as the linearizing scheme in solving the PDEs. The Figure (B.1), at the appendix, illustrates the flow chart of the implemented algorithm. The algorithm first, solves for the flow pressure implicitly. This new pressure value is used to update flow parameters in order to obtain the saturation explicitly. Afterwards, the algorithm proceeds to the next time-step to repeat the process.

3.6 Production Profile

Once the pressure and saturation profiles for both the similarity and numerical approaches are obtained, the Darcy's law is used to estimate the oil production rate. Under radial flow, oil rates can be expressed as:

$$q_o = A \frac{k k_{ro}}{\mu_o B_o} \frac{\partial p}{\partial r}\Big|_{r=r_w} \tag{3.64}$$

where all parameters are defined in the Nomenclature section. Equation (3.64) can however be re-written in terms of the Boltzmann variable as:

$$q_o = A \frac{k k_{ro}}{\mu_o B_o} \frac{1}{\sqrt{t}} \frac{dp}{d\xi}\Big|_{\xi=0} \tag{3.65}$$

The change of variable causes the derivative to be dependent on ξ. Evidently, the slow nature of the flow process is also accounted for; by the introduction of $\sqrt{t}$. This equation suggests that, the oil rate is inversely proportional to $\sqrt{t}$.

3.7 Analytical Solution for an Infnite Acting System

In this section, steps are undertaken to develop an analytical solution to the flow equations after the similarity transformations has been performed. The analytical solution is developed for a limiting case of short producing time at constant pressure production.

Recalling the ODEs resulting from the similarity transformation, that is;

$$\frac{d}{d\xi}\left(\alpha\frac{dp}{d\xi}\right) = -\frac{\xi}{2}\left(\frac{\partial\beta}{\partial p}\frac{dp}{d\xi} + \frac{\partial\beta}{\partial S_o}\frac{dS_o}{d\xi}\right) \tag{3.66}$$

$$\frac{dS_o}{d\xi} = -\frac{dp}{d\xi}\left[\frac{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial p} - \xi\left(R\frac{\partial\beta}{\partial p} - \frac{\partial b}{\partial p}\right)}{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial S_o} - \xi\left(R\frac{\partial\beta}{\partial S_o} - \frac{\partial b}{\partial S_o}\right)}\right] \tag{3.67}$$

The above equation are solved with the following boundary conditions:

$$p = p_w, \quad \xi = 0 \tag{3.68}$$

$$p = p_e, \quad \xi \to \infty \tag{3.69}$$

$$S_o = 1, \quad \xi \to \infty \tag{3.70}$$

Evidently, solving the ODEs in Equation (3.66) and Equation (3.67) requires initial and boundary conditions. Since Equation (3.66) is a second order ODE and Equation (3.67) is a first order ODE, two boundary conditions for pressure and an initial condition for saturation are sufficient to arrive at a solution. Under the assumption of an infinite acting reservoir, with constant pressure production at the wellbore, the conditions given by Equations (3.68), (3.69) and (3.70) apply.

Equation (3.68) indicates that, the pressure is constant and equal to p_w at the producing face, whereas Equation (3.69) and Equation (3.70) represent the pressure and saturation, respectively, as uniform initially (that is, at $t = 0$) and remain unchanged at the far boundary ($r \to \infty$).

3.7.1 Pressure Solution

The analytical solution for pressure is obtained by first substituting, Equation (3.67) into Equation (3.66) and evaluating the resulting equation for very large values of ξ ($\xi \to \infty$).

In order to reduce the non-linearity associated with Equation (3.66), the two phase pseudopressure(m) is introduced. This approach was adopted by several authors including Fetkovich (1973), Raghavan (1976) and Behmanesh (2016) in their study of reservoir flow behavior. In this study, a similar approach is used and the two phase pseudopressure is expressed as:

$$m(p) = \frac{1}{\alpha_i}\int_{p_b}^{p} \alpha dp = \frac{\mu_{oi}B_{oi}}{k_i k_{ro}^*}\int_{p_b}^{p} \frac{k k_{ro}}{\mu_o B_o} dp \tag{3.71}$$

where; $dm = \dfrac{\alpha}{\alpha_i} dp$.

Substituting Equation (3.71) into Equation (3.66) under the defined limiting assumption yields:

$$\frac{d^2 m}{d\xi^2} + \frac{\xi}{2\eta_\infty}\frac{dm}{d\xi} = 0 \tag{3.72}$$

where:

$$\eta_\infty = \frac{\alpha}{\dfrac{\partial \beta}{\partial p} + \dfrac{\partial \beta}{\partial S_o}\dfrac{dS_o}{dp}} \tag{3.73}$$

Replacing the full expressions for α and β into Equation (3.73) gives

$$\eta_\infty = \frac{k k_{ro}}{\phi \mu_o c_t^*}\frac{1}{f_o} \tag{3.74}$$

$$c_t^* = \frac{S_o B_o}{B_o}\frac{dR_s}{dp} - \frac{S_g}{B_g}\frac{dB_g}{dp} - \frac{S_o}{B_o}\frac{dB_o}{dp} + \frac{1}{\phi}\frac{d\phi}{dp} \tag{3.75}$$

$$f_o = \frac{1}{1 + \dfrac{k_{rg}}{k_{ro}}\dfrac{\mu_o}{\mu_g}} \tag{3.76}$$

Equation (3.74) is similar to the well known hydraulic diffusivity of a single phase reservoir, adjusted to reflect the effect of two phase flow and the evolution of gas out of the oil when the pressure drops below the bubble point pressure. Equation

(3.75) and Equation (3.76) on the other hand, account for the total compressibility of the fluids, and fractional flow of oil respectively.

Besides the mathematical derivation of Equation (3.75) and Equation (3.76), they imbibe varying physical interpretation. The different terms of Equation (3.75) are explained as follows:

1. The first term $\left(\left(\frac{S_o B_g}{B_o}\right)\frac{dR_s}{dp}\right)$ represents the amount of gas released per unit pore volume at reservoir conditions during the pressure drop of dp and is a positive quantity.
2. The second term $\left(\left(\frac{S_g}{B_g}\right)\frac{dB_g}{dp}\right)$ represents the effect of gas compressibility on the flow.
3. The third term $\left(\left(\frac{S_o}{B_o}\right)\frac{dB_o}{dp}\right)$ represents the effect of oil compressibility.
4. The last term $\left(\left(\frac{1}{\phi}\right)\frac{d\phi}{dp}\right)$ is the pore compressibility which may be considered as negligible in some simplifying cases.

The fractional flow equation, Equation (3.76), is akin to the conventional form of the Buckley Leverett fractional flow equation, which is employed, when oil displaces gas in a horizontal reservoir. This equation shows that, as the gas mobility $\left(\frac{k_{rg}}{\mu_g}\right)$ decreases, f_o increases. Consequently, when f_o increases, the term $c_t^* f_o$ which represents the energy of the reservoir, increases which implies an increased oil recovery.

However, it should be duly noted that, evaluating the two phase pseudopressure, m requires the relationship between saturation and pressure. Tabatabaie and Pooladi-Darvish (2016) in their work on multiphase flow showed that Equation (3.77) can be used to approximate the saturation pressure relationship for cases of this nature.

$$S_o(p) = 1 + \mu_{oi} B_{gi} \left(\frac{dR_s}{dp} \right)_i \int_{p_i}^{p} \frac{1}{\mu_o B_o} dp \tag{3.77}$$

Equation (3.77) offers independence from absolute permeability and relative permeability curves. Therefore, the right hand side of Equation (3.77) presents a sole function of pressure. Thus the saturations of fluids under prevailing pressures can be readily evaluated. Several attempts have also been established in literature to obtain a function for saturation in terms of pressure. For example: Raghavan (1976) suggested using the producing GOR to establish the saturation-pressure relationship. Behmanesh (2016) developed an alternative solution to establish the saturation-pressure relationship, and they solved it numerically. The formulation presented in Equation (3.77) is direct and does not require a numerical solution.

At this point, an analytical solution of Equation (3.72) is possible if the variation of η_∞ with m is considered negligible. This assumption renders Equation (3.72) into a linear ODE, which for constant pressure production can be solved together with the boundary conditions in Equation (3.68) and Equation (3.69) to obtain:

$$m(\xi) = (m_i - m_w) Erfc\left(\frac{\xi}{2\sqrt{\eta_\infty}} \right) \tag{3.78}$$

where m_i and m_w are two phase pseudopressures evaluated at the initial and flowing pressure respectively. Assuming that all terms in η_∞ are constant and equal to their initial values, $\eta_{\infty i}$, Equation (3.78) becomes:

$$m(\xi) = (m_i - m_w) Erfc\left(\frac{\xi}{2\sqrt{\eta_{\infty i}}} \right) \tag{3.79}$$

3.7.2 Saturation Solution

Next, recalling Equation (3.67), an analytical solution for saturation is developed under the limiting assumption.

$$\frac{dS_o}{d\xi} = -\frac{dp}{d\xi}\left[\frac{2\alpha\dfrac{dp}{d\xi}\dfrac{\partial R}{\partial p} - \xi\left(R\dfrac{\partial \beta}{\partial p} - \dfrac{\partial b}{\partial p}\right)}{2\alpha\dfrac{dp}{d\xi}\dfrac{\partial R}{\partial S_o} - \xi\left(R\dfrac{\partial \beta}{\partial S_o} - \dfrac{\partial b}{\partial S_o}\right)}\right] \tag{3.80}$$

For large values of ξ ($\xi \to \infty$), Equation (3.80) is simplified with the two phase pseudopressure defined by $dm = \dfrac{\alpha}{\alpha_i}dp$ to the equation below:

$$\frac{dS_o}{d\xi} = \frac{\alpha_i\left(R\dfrac{\partial \beta}{\partial p} - \dfrac{\partial b}{\partial b}\right)}{\alpha\left(R\dfrac{\partial \beta}{\partial S_o} - \dfrac{\partial b}{\partial S_o}\right)}\frac{dm}{d\xi} \tag{3.81}$$

Incorporating the full expressions of the parameters, α, β, R and b into Equation (3.80) results into Equation (3.81) given below:

$$\frac{dS_o}{d\xi} = \frac{k_i k_{ro}^*}{\mu_{oi}B_{oi}}\frac{B_o\mu_o}{kk_{ro}}c_{so}^*\frac{dm}{d\xi} \tag{3.82}$$

where;

$$c_{so}^* = c_t^* f_o - c_{ox}S_o \tag{3.83}$$

$$c_{ox} = \frac{1}{\phi}\frac{d\phi}{dp} - \frac{1}{B_o}\frac{dB_o}{dp} \tag{3.84}$$

$$N = \frac{k_i k_{ro}^*}{\mu_{oi}B_{oi}}\frac{\dfrac{B_g}{B_o}\dfrac{dR_s}{dp} + \dfrac{k_{rg}}{k_{ro}}\dfrac{B_g}{B_o}\dfrac{d}{dp}\left(\dfrac{\mu_o}{\mu_g}\dfrac{B_o}{B_g}\right)}{\dfrac{kk_{ro}}{\mu_o B_o}}f_o(m_i - m_w) \tag{3.85}$$

Evaluating the coefficients of ($\dfrac{dm}{d\xi}$) of Equation (3.82) at their initial values (since $\xi \to \infty$) yields:

$$\frac{dS_o}{d\xi} = \frac{N_i}{m_i - m_w}\frac{dm}{d\xi} \tag{3.86}$$

where;

$$N_i = \left(\frac{B_g}{B_o}\frac{dR_s}{dp}\right)_i (m_i - m_w) \tag{3.87}$$

The derivative $\dfrac{dm}{d\xi}$ of Equation (3.82) can be derived by differentiating the pressure solution obtained earlier. Since Equation (3.82) is a first order ODE, it requires a single boundary condition for saturation Equation (3.70) to solve it.

Solving the ODE, Equation (3.86) with the appropriate boundary condition Equation (3.70) gives:

$$S_o(\xi) = 1 - N_i Erfc\left(\frac{\xi}{2\sqrt{\overline{\eta}_\infty}}\right) \tag{3.88}$$

The full process of developing the analytical solution is presented at the Appendix.

3.7.3 Production Parameters

The derived pressure and saturation solution are utilized to derive equations for oil production and cumulative production.

The cumulative oil production is expressed as:

$$N_p(t) = \int_0^t q_o dt \tag{3.89}$$

where q_o, according to Darcy's equation, is given by

$$q_o = A\frac{kk_{ro}}{\mu_o B_o}\frac{\partial p}{\partial r}\Big|_{r=r_w} \tag{3.90}$$

Equation (3.90) is written in terms of the two phase pseudopressure as:

$$q_o = A\frac{kk_{ro}}{\mu_{oi} B_{oi}}\frac{\partial m}{\partial \xi}\Big|_{\xi=0} \tag{3.91}$$

Inserting the derivative of the pseudopressure (Equation (3.79)) yields:

$$q_o = A\frac{kk_{ro}}{\mu_{oi} B_{oi}}\left(\frac{m_i - m_w}{\sqrt{\pi\overline{\eta}_\infty}}\right)\frac{1}{\sqrt{t}} \tag{3.92}$$

Therefore the cumulative oil production given by Equation (3.89) is expressed as:

$$N_p(t) = A\frac{k_i k_{ro}^*}{\mu_{oi} B_{oi}} \left(\frac{m_i - m_w}{\sqrt{\pi \overline{\eta}_\infty}} \right) \sqrt{t} \tag{3.93}$$

3.8 Determination of Gas-Oil Ratio

The similarity transformation affords a means to express saturation as a unique function of pressure. Under the constant production pressure scenario, a constant saturation is imposed at the sandface or production face.

Studies have shown that, although during transient flow, the average pressure and saturation within the region of depletion are constant, the constant producing gas-oil ratio (GOR) typical of tight oil reservoir cannot be attributed to the constant average properties within the depletion zone. Based on this, the GOR is a function of the conditions at the sandface. According to Tabatabaie and Pooladi-Darvish (2016), the GOR for tight oil reservoirs can be evaluated by Equation (3.94):

$$GOR = R_s + \frac{k_{rg} \mu_o B_o}{k_{ro} \mu_g B_g} \tag{3.94}$$

Chapter 4

Results and Discussion

4.1 Introduction

In this section, an analysis is performed on a base case to study the behavior of the reservoir during transient radial flow when it is subject to constant pressure production.

4.2 Data Simulation

A cylindrical reservoir of radius(r), $800m$ and pay thickness(H), $50m$, is considered. The porosity and the initial permeability of the flow problem are taken to be 0.1 and $0.01md$ respectively. The flow model is initially saturated with oil at a saturation pressure of $50000kPa$ and produced at constant flowing pressure of $10000kPa$. The fluid properties of the base case are presented in Figure 4.1 to Figure 4.5 below. The Corey-type relative permeability functions defined under Equation (4.1) and Equation (4.2) are employed to relate the variation of relative permeabilities to the saturation. This is illustrated by Figure (4.6).

$$k_{ro} = k_{ro}^* S_o^{n_o} \tag{4.1}$$

$$k_{rg} = k_{rg}^* (1 - S_o)^{n_g} \tag{4.2}$$

Figures (4.1) through to (4.6) describe the Pressure-Volume-Temperature (PVT) parameters that are adopted for the study. Figure (4.1), for example, illustrates a typical behavior of the oil formation volume factor. Below the bubble point pressure, the oil formation volume factor increases with pressure. This is because more gas goes into solution as the pressure is increased causing the oil to swell.

Due to the dramatically different conditions prevailing at the reservoir when compared to the conditions at the surface, it is not expected that 1 barrel of fluid at reservoir conditions should contain the same amount of matter as 1 barrel of fluid at surface conditions. The volumetric factors (B_o) and (B_g) are introduced in the calculations in order to readily relate the volume of fluids that are obtained at the surface (stock tank) to the volume that the fluid actually occupy when it is compressed in the reservoir.

Figure (4.5), which illustrates the solution gas-oil ratio to pressure is an integral parameter in the study of tight oil reservoirs. It increases approximately linearly with pressure and is a function of the oil and gas composition. Tight oils contains high amounts of dissolved gas hence the solution gas-oil ratio increases with pressure as observed in the Figure (4.5) until the bubble point pressure is reached, after which it is a constant, and the oil is said to be undersaturated.

The end point relative permeabilities are considered as 1 whereas the gas and oil relative permeability exponents are considered as 2. This data is obtained from the work of Tabatabaie and Pooladi-Darvish (2016)

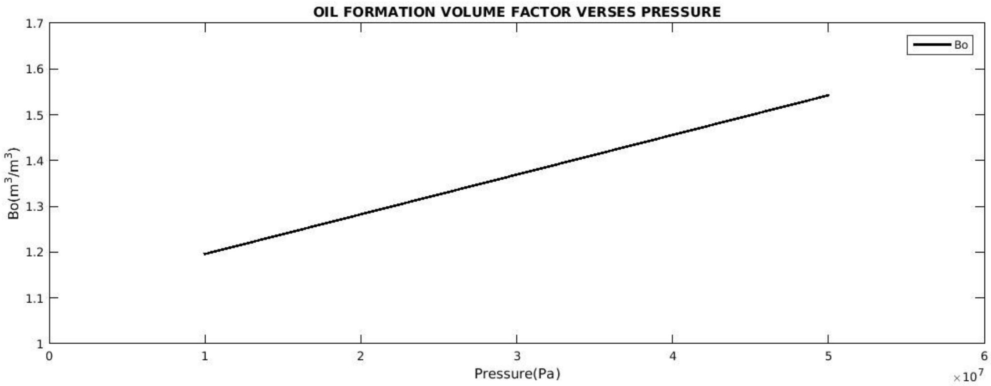


Figure 4.1: Oil formation volume factor verses pressure

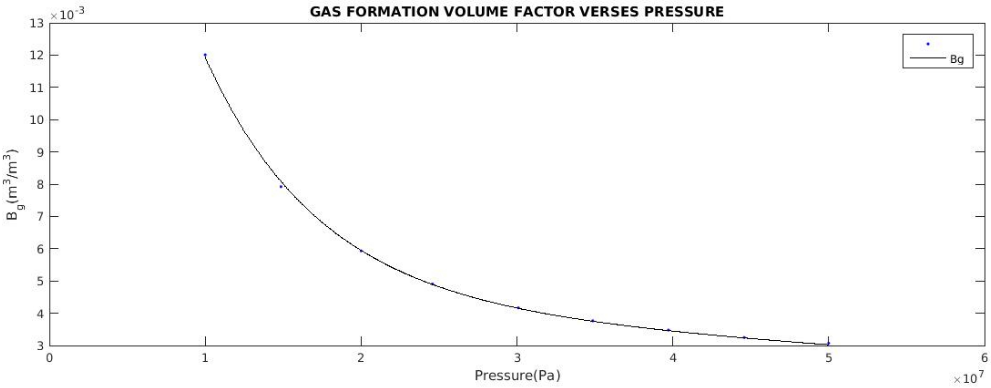


Figure 4.2: Gas formation volume factor verses pressure

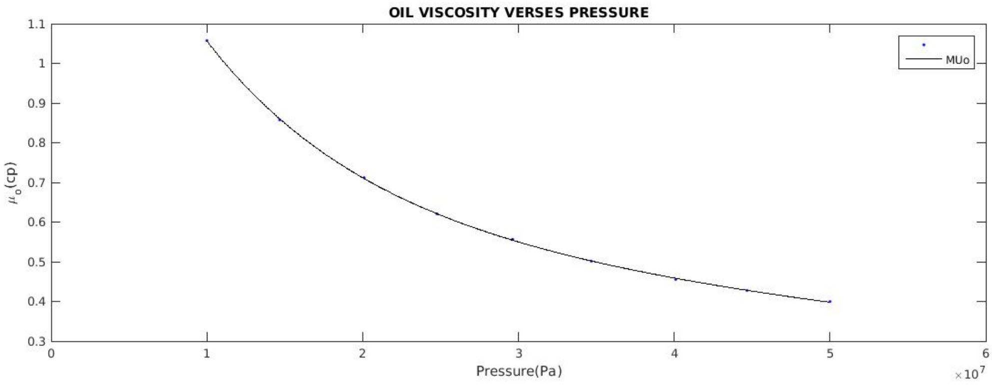


Figure 4.3: Oil viscosity verses pressure

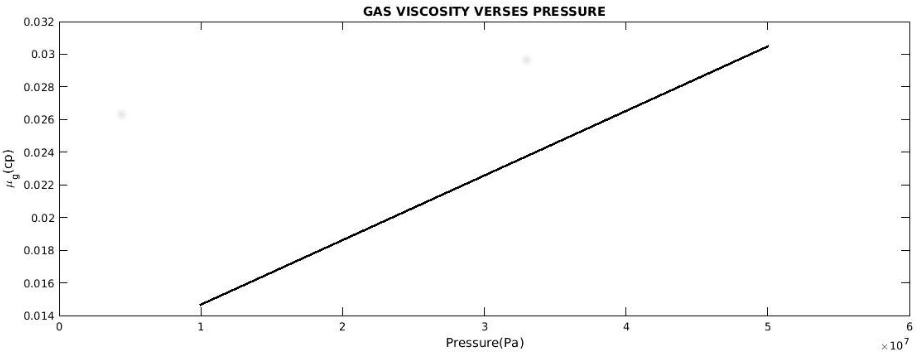


Figure 4.4: Gas viscosity verses pressure

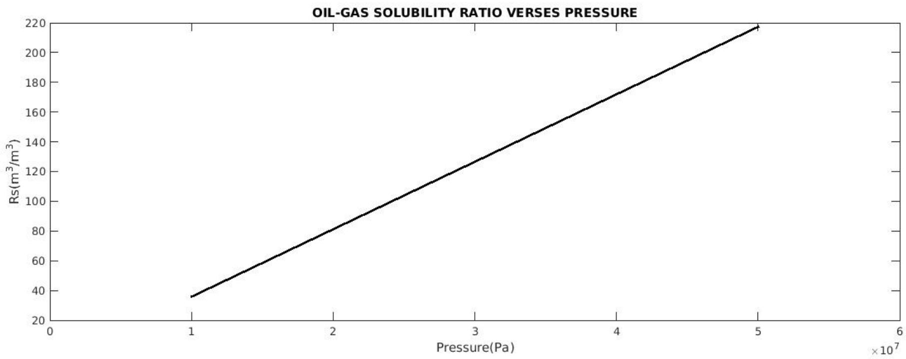


Figure 4.5: Oil-Gas solubility ratio verses pressure

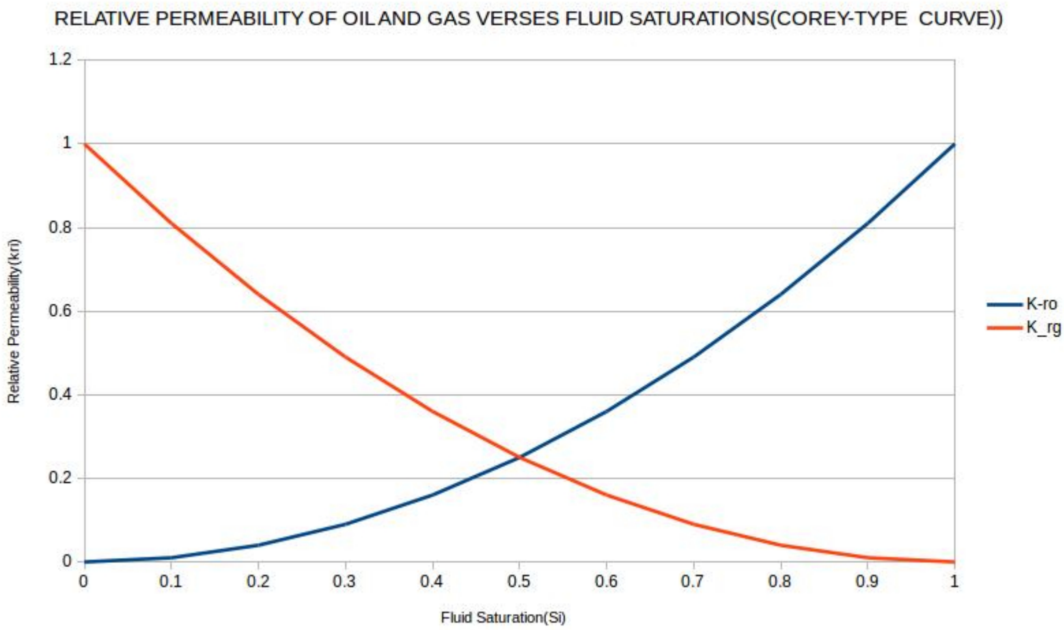


Figure 4.6: Corey-type relative permeability plots for oil and gas

4.2.1 Results of Simulations

The model exhibits transient flow over the given period of time after the pressure disturbance has been created in the reservoir. The reduction of reservoir pressure at initial conditions to the constant pressure production at the well-bore or production face causes reservoir fluids to flow near the vicinity of the well. The pressure drop of the expanding fluid will provoke flow from further, undisturbed regions in the reservoir. The pressure disturbance and fluid movement will continue to propagate radially away from the well-bore over the given period.

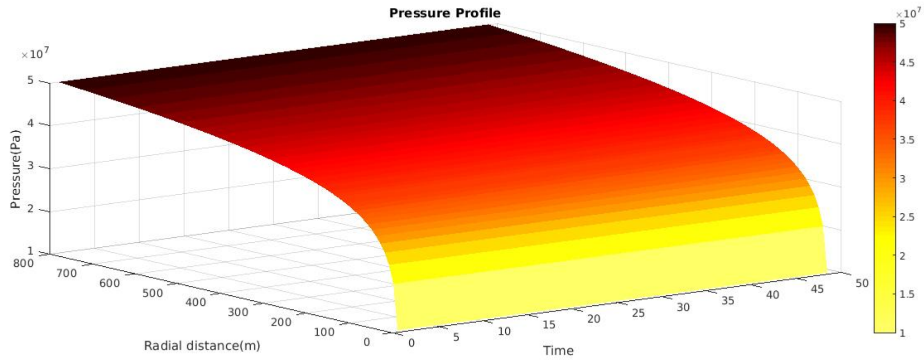


Figure 4.7: Pressure profile in time and space

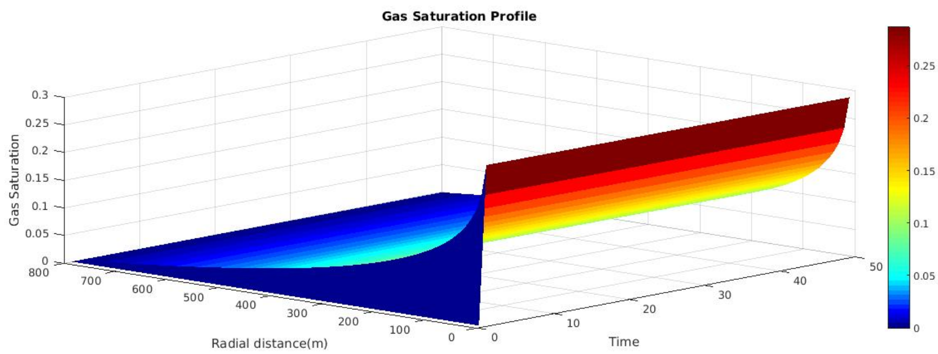


Figure 4.8: Gas saturation profile in time and space

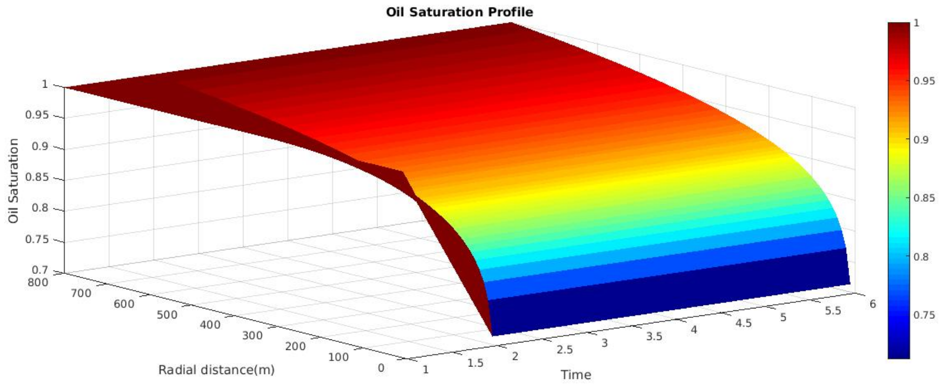


Figure 4.9: Oil saturation profile in time and space

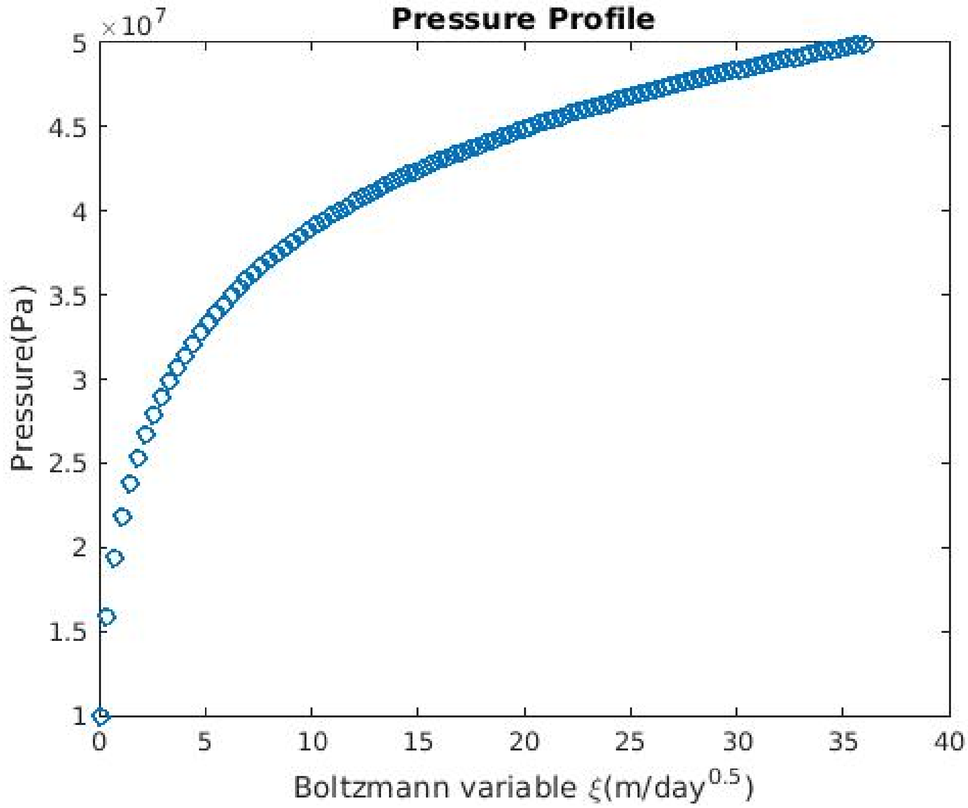


Figure 4.10: Pressure profile in in terms of ξ

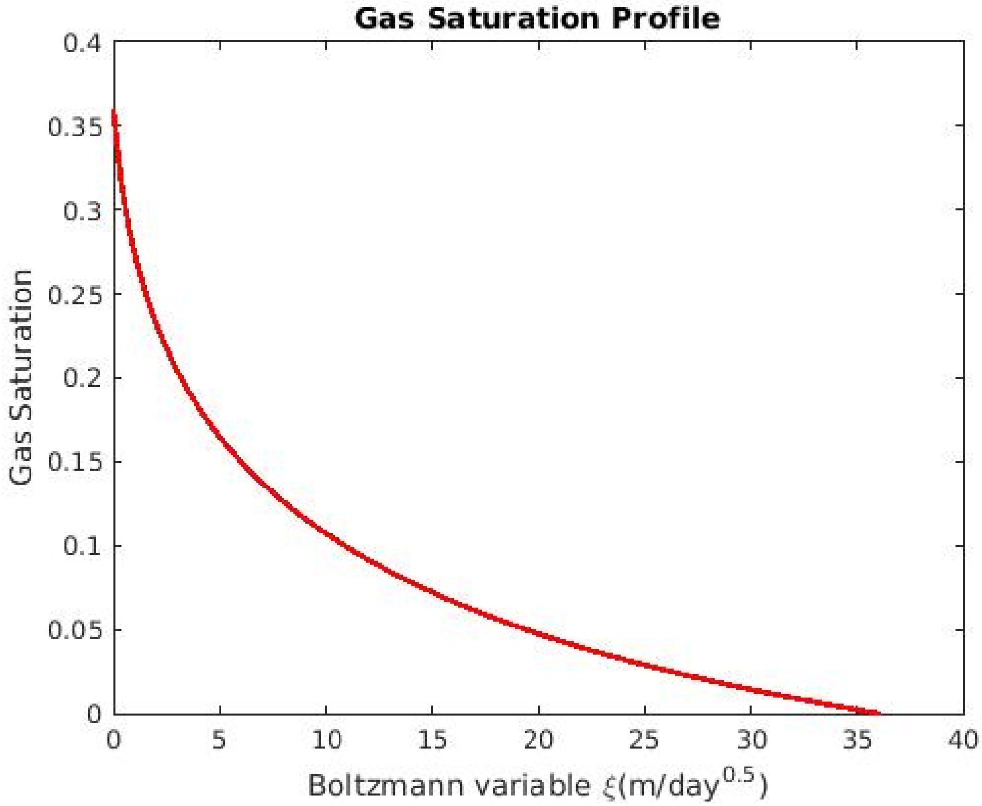


Figure 4.11: Gas saturation profile in in terms of ξ

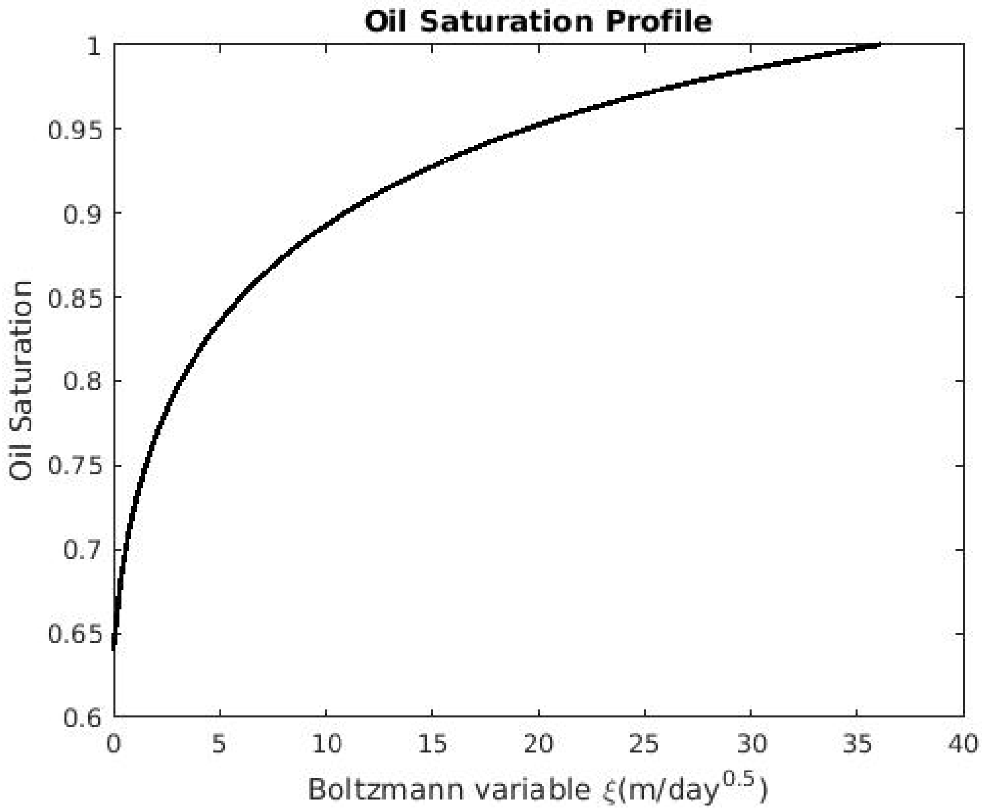
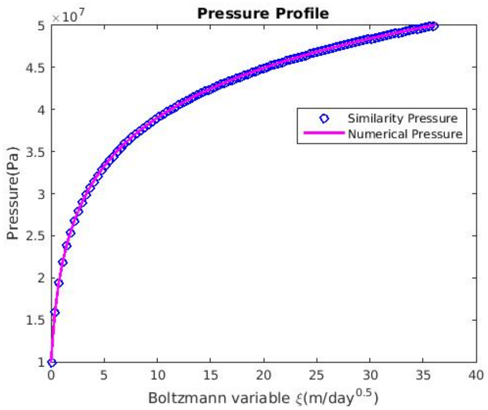
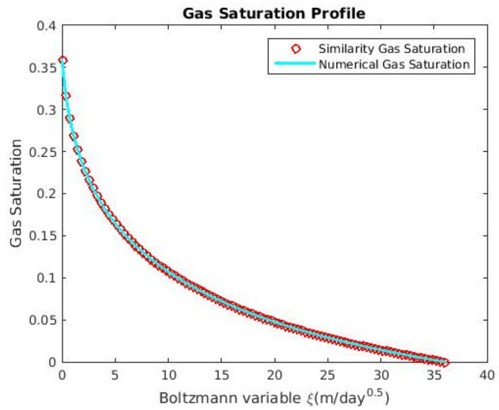


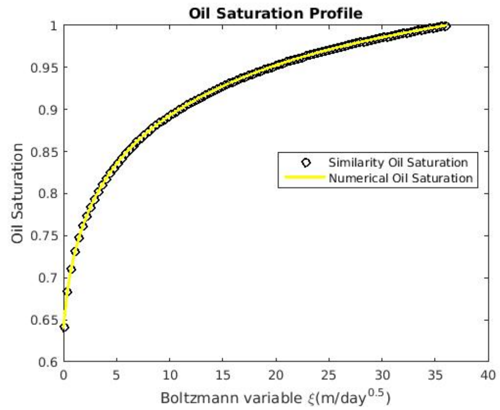
Figure 4.12: Oil saturation profile in in terms of ξ



(a) Pressure profile from numerical approach



(b) Pressure profile from similarity approach



(c) Pressure profile from similarity approach

Figure 4.13: Comparison of pressure profiles resulting from the numerical solution and similarity solution

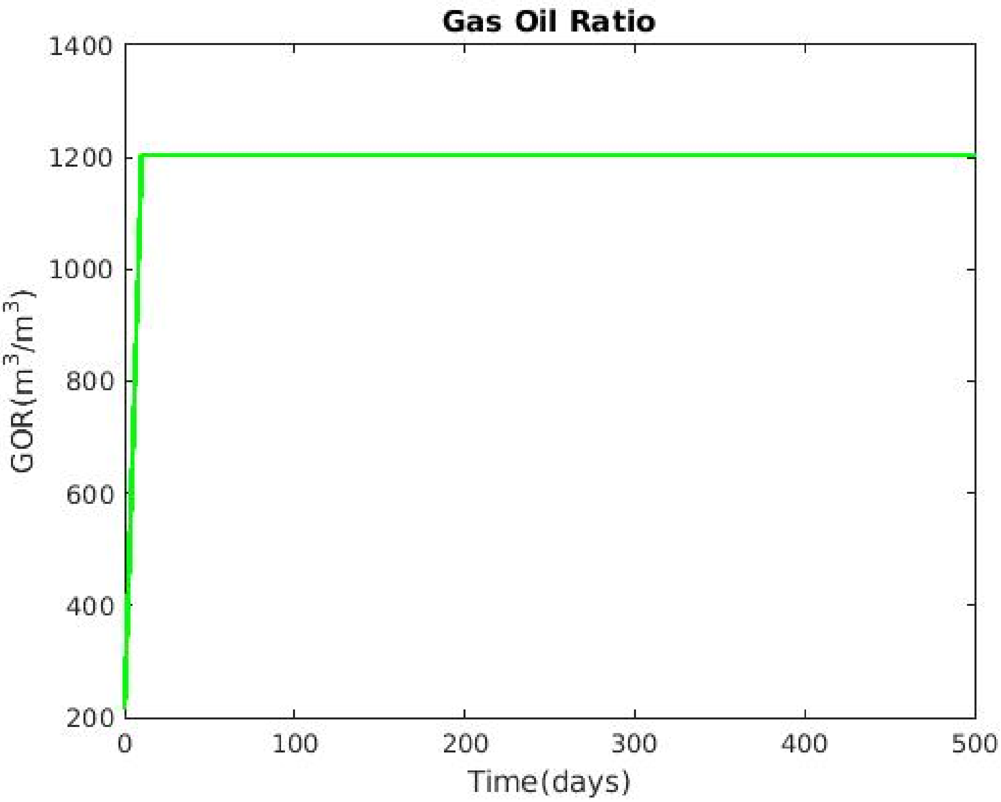


Figure 4.14: The Gas-Oil Ratio pattern for the entire simulation period

4.2.2 Observations

The constant pressure production of $10000kPa$ causes a two phase flow of oil and gas through the porous media into the well bore. The drop of pressure below the bubble point of pressure($50000kPa$) causes the evolution of gases out of solution as flow propagates into the well bore. The nature of the pressure drop over the domain directly affects the saturation distribution of fluids (oil and gas) over the domain.

In the time for which the transient condition is applied it is observed that, the pressure profile for each time step do not significantly differ. This is as result to the assumption made on the proportion of oil to gas in the system. The slightly compressible nature of the system causes the system to behave in a manner similar to a single phase flow.

Figures (4.7), (4.9) and (4.8) present the pressure and saturation (oil and gas) distributions respectively, over the period of transient two-phase flow regime. These distributions over the domain in time and space result from the numerical approach for solving the PDEs that describe the flow process. These plots show that, as time increases, pressure disturbance propagates throughout the reservoir. After the pressure drops below the bubble point pressure (which, in this case, is equal to the initial reservoir pressure), gas is released, and the oil saturation reduces.

It is also observed that the solution found by solving ODEs in the similarity approach do not significantly differ from the profiles obtained by solving the original PDEs numerically. These are also illustrated in Figures (4.13a), (4.13b) and (4.13c).

Last but not least, Figure (4.14) shows that, for the data considered, the producing GOR is about four times larger than the initial GOR of the reservoir. The producing GOR maintains a constant value throughout the period of the constant pressure production at the sandface.

4.3 Discussion

In this study, the highly non-linear radial flow equations governing oil and gas flow in tight oil reservoir producing at constant pressure are converted to two nonlinear ODEs for pressure and saturation. Obtaining solutions to the resulting non-linear ODEs after the Boltzmann transformation is simpler and much faster than solving the original PDEs.

The transformation of the PDEs to the ODEs was dependent on the condition that, all independent variables (r and t) can be combined to a single form of the

Boltzmann variable (ξ). In this way, none of the original independent variables of the mathematical model, remained after the transformation was performed. This study showed that, during transient radial flow period under constant bottom-hole pressure production, pressure and saturation are unique functions of the Boltzmann variable.

Since, the PDEs describing the problem are in terms of radius, r and time, t are reduced to ODEs in terms of the Boltzmann variable, ξ, it causes the transient pressure and saturation profiles in real time domain to collapse unto a single curve when plotted verses the Boltzmann variable, ξ as observed in the Figures (4.13a), (4.13b) and (4.13c). However, all the real time solutions ($p(r,t)$ or $S_o(r,t)$) can be readily calculated by taking any point on the plot of $p(\xi)$ (or $S_o(\xi)$ and assigning their values to a corresponding distance found from $\xi = \dfrac{r}{\sqrt{t}}$ at any particular time.

The model framework provides the opportunity to explain some of the observed behavior of tight oil reservoirs such as the constant producing GOR during transient radial flow of a reservoir producing at constant pressure. The producing GOR is controlled by the pressure and saturation at the sandface or production face of the reservoir. This behavior is not attributed to the average properties within the region of depletion. It is shown by Figure (4.14) that, if two phase fluid flow can be modeled with Equation (3.1) and Equation (3.1), the producing GOR is constant during the transient flow at constant producing pressure. If any of the assumptions made in the development of the model are not satisfied, the producing GOR may vary with time.

Figure (4.14) shows that the instantaneous producing gas-oil ratio (GOR) is constant during transient radial flow for constant pressure production. The constant behavior of producing GOR has also been observed by several researchers, including Whitson and Sunjerga (2012) and Behmanesh (2016). This affirms the typical behavior of tight oil reservoirs when subject to constant pressure at the producing face.

Equation (3.94) gives the plot shown in Figure (4.14) which shows that, during transient radial flow at constant pressure production, the producing GOR is a function of fluid properties, relative permeability and flowing pressure. Figure (4.14) also shows that, for the data considered, the producing GOR is about four times larger than the initial solution GOR. This implies that the recombination of fluid samples collected at the surface in the ratio of producing GOR does not represent the in-situ reservoir fluid.

It is based on the outcomes of the similarity and numerical approaches that fostered the determination of an analytical solution for the limiting case (infinite acting conditions). The viability of this endeavour leans on the mathematical fact that, radial flow under an infinite acting boundary condition eventually takes a linear flow pattern for larger reservoir extents. In the development of the analytical solution, the infinite acting boundedness imposed on the resevoir caused the radial flow regime to take a linear flow pattern under the limiting assumption. Hence, the analytical solution developed in this work conformed with the results obtained by Tabatabaie and Pooladi-Darvish (2016), in his work on linear flow in tight oil reservoirs.

Despite this development, the practicality of the results of the analytical solution remains stalled. From, Equations (3.78) through to (3.93), the initial gas saturation is zero due to the infinite boundary condition imposed on the flow problem. Hence, evaluation of η_∞ at initial conditions removes the effect of gas mobility from the analytical solution (since, $\left(\dfrac{k_{rg}}{\mu_g}\right)_i = 0$). However, it accounts for the effect of gas on oil flow, by changing k_{ro}, and the effect of gas evolution on compressibility, given as, $\left(\frac{S_oB_g}{B_o}\right)_i \left(\frac{dR_s}{dp}\right)_i$.

In the methodology, it was explained that, reservoirs with higher gas mobility lose energy support faster than reservoirs with lower gas mobility and as such produces less. It is therefore necessary to take into account the effect of gas mobility in Equation (3.92) and (3.93) by determining a correction factor.

It is worth noting that, no assumptions regarding the variation of porosity and absolute permeability with pressure were considered in the radial flow model. The saturation-pressure relationship as well as the relation for the producing GOR used in this study are independent of absolute permeability. The total system compressibility derived in this study on two-phase-flow is similar to the formulations introduced by Martin (1959). As discussed by Ramey (1964), the total system compressibility, rather than the single phase compressibility, is more useful for multiphase pressure transient analysis.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

The radial multiphase flow phenomenon in tight oil reservoirs presents itself as a nonlinear problem yet the equations used in this study provide a reasonable explanation of the physics in the reservoir. It took into account the occurrence of both dissolved and evolved gases in the system.

This study presented the application of the similarity transformation to obtain ODEs that have the capability of rigorously solving the governing PDEs for the radial two phase flow in tight oil reservoir. The simplicity of the system of ODEs presented here, enabled the quick calculation of reservoir profiles for the problem under consideration.

The similarity approximation which facilitated the analyses of the problems was adequate. For the radial flow problem defined in this study, the similarity approximation delivered a good estimate in comparison to the numerical solution. The similarity solutions are satisfactorily sufficient in solving the flow problem.

The results from the producing GOR concludes that, the recombination of fluid samples collected at the surface in the ratio of producing GOR does not represent the in-situ reservoir fluid. This is a typical behavior of tight oil reservoirs which are produced at constant pressure.

The analytical solution developed in this work conforms with the results obtained by Tabatabaie and Pooladi-Darvish (2016), in his work on linear flow in tight oil reservoirs.

Despite the somewhat ideal assumptions impose on the model, the variability of the data used and the concepts adopted helped to theoretically bring the problem representation closer to similar cases in literature. This was achieved as the constant GOR which is a common phenomenon in tight oil reservoir studies was attained.

5.2 Recommendations

It is recommended that strict adherence to the assumptions on the model be considered when carrying out this work. This study was achieved under many simplifying assumptions to the flow. It is necessary to take into account the full mechanics of the problem to better understand the viability of similarity transformation in reservoir engineering techniques. Since, no assumptions regarding the variation of porosity and absolute permeability with pressure were considered in this study, it is recommended that, further investigation be carried out on this factor.

The practicality of the analytical solution remains stalled. Due to the infinite acting boundary assumption imposed on the flow, the initial gas saturation was considered as zero which ignored the effect of gas mobility from the analytical solution. It is therefore necessary to develop a correction factor to account for the evolution of gases when using the analytical solution. This will provide better insight to the behavior of tight oil reservoirs exhibiting multiphase flow.

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Appendix A

Analytical Solution as $\xi \to 0$

In this section, an analytical solutions is developed for the pressure and saturation by evaluating the terms of the transformed diffusivity equations at large values of ξ. Practically, this corresponds to small values of time or large values of distance, since ξ is a function of r and t. First, a pressure and saturation equation must be obtained

A.1 Pressure Equation

The oil diffusivity equation is expressed as:

$$\frac{d}{d\xi}\left(\alpha\frac{dp}{d\xi}\right) + \frac{\alpha}{\xi}\frac{dp}{d\xi} = -\frac{\xi}{2}\frac{\partial\beta}{\partial\xi} \tag{A.1}$$

At large values of ξ, $\dfrac{\alpha}{\xi}\dfrac{\partial p}{\partial\xi} \to 0$.
Since β is a function of p and So, Equation (A.1) is expanded and written as:

$$\frac{d}{d\xi}\left(\alpha\frac{dp}{d\xi}\right) = -\frac{\xi}{2}\left[\frac{\partial\beta}{\partial p}\frac{dp}{d\xi} + \frac{\partial\beta}{\partial So}\frac{dSo}{d\xi}\right] \tag{A.2}$$

Equation (A.2) has two unknowns (pressure and saturation). If saturation were known, Equation (A.2) could be solved directly to find pressure as a function of the Boltzmann variable. However, saturation profile is not known at priori. In order to complete the system of equations and unknowns, the diffusivity equation of gas is employed.

A.2 Saturation Equation

The gas diffusivity equations are expressed as:

$$\alpha \frac{\partial R}{\partial \xi}\frac{dp}{d\xi} = \frac{\xi}{2}\left[R\frac{\partial \beta}{\partial \xi} - \frac{\partial b}{\partial \xi}\right] \tag{A.3}$$

Since R, b and β are a function of p and So, Equation (A.3) is expanded and written as:

$$\alpha\left(\frac{\partial R}{\partial p}\frac{dp}{d\xi} + \frac{\partial R}{\partial So}\frac{So}{d\xi}\right)\frac{dp}{d\xi} = \frac{\xi}{2}\left[R\left(\frac{\partial \beta}{\partial p}\frac{dp}{d\xi} + \frac{\partial \beta}{\partial So}\frac{So}{d\xi}\right) - \left(\frac{\partial b}{\partial p}\frac{dp}{d\xi} + \frac{\partial b}{\partial So}\frac{So}{d\xi}\right)\right] \tag{A.4}$$

Re-arranging Equation (A.4) yields the saturation equation, given as:

$$\frac{dS_o}{d\xi} = -\frac{dp}{d\xi}\left[\frac{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial p} - \xi\left(R\frac{\partial \beta}{\partial p} - \frac{\partial b}{\partial p}\right)}{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial S_o} - \xi\left(R\frac{\partial \beta}{\partial S_o} - \frac{\partial b}{\partial S_o}\right)}\right] \tag{A.5}$$

A.3 Boundary Conditions

To sovle the pressure and saturation equations, the following initial and boundary conditions are used;

$$p = p_w, \quad \xi = 0 \tag{A.6}$$

$$p = p_e, \quad \xi \to \infty \tag{A.7}$$

$$S_o = 1, \quad \xi \to \infty \tag{A.8}$$

A.4 Pressure Solution

Obtaining the analytical solution for pressure requires substitution of Equation (A.5) into Equation (A.2) to obtain:

$$\frac{d}{d\xi}\left(\alpha\frac{dp}{d\xi}\right)=\frac{\xi}{2}\left(\frac{\partial\beta}{\partial p}\frac{dp}{d\xi}+\frac{\partial\beta}{\partial S_o}\frac{dp}{d\xi}\left[\frac{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial p}-\xi\left(R\frac{\partial\beta}{\partial p}-\frac{\partial b}{\partial p}\right)}{2\alpha\frac{dp}{d\xi}\frac{\partial R}{\partial S_o}-\xi\left(R\frac{\partial\beta}{\partial S_o}-\frac{\partial b}{\partial S_o}\right)}\right]\right) \tag{A.9}$$

The two-phase pseudopressure,(m) given by

$$m(p)=\frac{1}{\alpha_i}\int_{p_b}^{p}\alpha dp=\frac{\mu_{oi}B_{oi}}{k_i k^*_{ro}}\int_{p_b}^{p}\frac{kk_{ro}}{\mu_o B_o}dp \tag{A.10}$$

is introduced to reduce the non-linearity associated with Equation (A.9) . Here, the two-phase pseudopressure is defined as $dm=\frac{\alpha}{\alpha_i}dp$ which yields:

$$\frac{d^2m}{d\xi^2}=\frac{\xi\alpha_i}{2\alpha}\left[\frac{\partial\beta}{\partial p}-\frac{\partial\beta}{\partial S_o}\left(\frac{R\frac{\partial\beta}{\partial p}-\frac{\partial b}{\partial p}}{R\frac{\partial\beta}{\partial S_o}-\frac{\partial b}{\partial S_o}}\right)\right]\frac{dm}{d\xi} \tag{A.11}$$

Under the assumption of $\xi\to\infty$ Equation (A.11) is expressed as:

$$\frac{d^2m}{d\xi^2}+\frac{\xi}{2\eta_\infty}\frac{dm}{d\xi}=0 \tag{A.12}$$

where;

$$\eta_\infty=\frac{\alpha\left(R\frac{\partial\beta}{\partial S_o}-\frac{\partial b}{\partial S_o}\right)}{\frac{\partial\beta}{\partial p}+\frac{\partial\beta}{\partial S_o}\frac{dS_o}{dp}} \tag{A.13}$$

Replacing α, β, b and R into Equation (A.13) gives

$$\eta_\infty=\frac{kk_{ro}}{\phi\mu_o c^*_t}\frac{1}{f_o} \tag{A.14}$$

where;

$$c_t^* = \frac{S_o B_o}{B_o}\frac{dR_s}{dp} - \frac{S_g}{B_g}\frac{dB_g}{dp} - \frac{S_o}{B_o}\frac{dB_o}{dp} + \frac{1}{\phi}\frac{d\phi}{dp} \tag{A.15}$$

$$f_o = \frac{1}{1 + \frac{k_{rg}}{k_{ro}}\frac{\mu_o}{\mu_g}} \tag{A.16}$$

Now, the process for obtaining the analytical solution of Equation (A.12) during constant pressure production is carried out. Under the assumption that η_∞ is a weak function of m, ($\overline{\eta}_\infty = \eta_\infty = constant$) the solution of Equation (A.12) along using two boundary conditions for pressure (Equation (A.6) and Equation (A.7)) is readily obtained.

A.4.1 Solving for $m(\xi)$

From Equation (A.12), given by:

$$*\frac{d^2 m}{d\xi^2} + \frac{\xi}{2\eta_\infty}\frac{dm}{d\xi} = 0 \tag{A.17}$$

Reducing the order of Equation (A.17) by letting $\frac{dm}{d\xi} = u$, then Equation (A.17) becomes:

$$\frac{du}{d\xi} = -\frac{\xi}{2\eta_\infty}u \tag{A.18}$$

By the method of separation of variables and integrating both sides the following equation results:

$$u = e^{-\frac{\xi^2}{4\eta_\infty}}.c_1 \tag{A.19}$$

where c_1 is a constant of integration. Substituting u (Equation (A.19)) back into $\frac{dm}{d\xi} = u$ and integrating both sides gives;

$$m(\xi) = c_i Erfc\left(-\frac{\xi}{2\sqrt{\eta_\infty}}\right) + c_2 \tag{A.20}$$

Given that, the special values $Erfc(0) = 1$ and $Erfc(\infty) = 0$. Evaluation $m(\xi)$ at $\xi = 0$ and $\xi = \infty$ shows that

$$m(\xi = 0) = m_w = c_1 + c_2$$

and

$$m(\xi = \infty) = m_i = c_2$$

Hence, the pressure solution becomes;

$$m(\xi) = m_i - (m_i - m_w)Erfc\left(\frac{\xi}{2\sqrt{\eta_\infty}}\right) \tag{A.21}$$

It should be noted that the evaluation of pseudopressure, in Equation (A.21) , requires a knowledge of the saturation-pressure relationship. Since this relationship is not known priori, it is alternatively convenient to explore a solution where pseudopressure is evaluated at initial oil saturation.

A.5 Saturation Solution

For large values of ξ ($\xi \to \infty$) Equation (A.5) is simplified by introducing the two phase pseudo-pressure defined as $dm = \frac{\alpha}{\alpha_i}dp$. This yields;

$$\frac{dS_o}{d\xi} = \frac{\alpha_i \left(R\frac{\partial \beta}{\partial p} - \frac{\partial b}{\partial b}\right)}{\alpha \left(R\frac{\partial \beta}{\partial S_o} - \frac{\partial b}{\partial S_o}\right)} \frac{dm}{d\xi} \tag{A.22}$$

Incorporating the parameters, α, β, R and b into Equation (A.22) results to:

$$\frac{dS_o}{d\xi} = \frac{k_i k_{ro}^*}{\mu_{oi} B_{oi}} \frac{B_o \mu_o}{k k_{ro}} c_{to}^* \frac{dm}{d\xi} \tag{A.23}$$

where;

$$c_{so}^{*} = c_t^{*} f_o - c_{ox} S_o \tag{A.24}$$

$$c_{ox} = \frac{1}{\phi}\frac{d\phi}{dp} - \frac{1}{B_o}\frac{dB_o}{dp} \tag{A.25}$$

$$N = \frac{k_i k_{ro}^{*}}{\mu_{oi} B_{oi}} \frac{\dfrac{B_g}{B_o}\dfrac{dR_s}{dp} + \dfrac{k_{rg}}{k_{ro}}\dfrac{B_g}{B_o}\dfrac{d}{dp}\left(\dfrac{\mu_o}{\mu_g}\dfrac{B_o}{B_g}\right)}{\dfrac{k k_{ro}}{\mu_o B_o}} f_o (m_i - m_w) \tag{A.26}$$

Evaluating the coefficients of Equation (A.23) at their initial values (for $\xi \to \infty$) yields:

$$\frac{dS_o}{d\xi} = \frac{N_i}{m_i - m_w}\frac{dm}{d\xi} \tag{A.27}$$

where;

$$N_i = \left(\frac{B_g}{B_o}\frac{dR_s}{dp}\right)_i (m_i - m_w) \tag{A.28}$$

Since Equation (A.27) is a first order ODE, a single initial condition for saturation given by Equation (A.8) is sufficient to solve it.

A.5.1 Solving for $So(\xi)$

Recall the derivative $\frac{dm}{d\xi}$ of the pressure solution (Equation (A.19)) as:

$$u = e^{-\frac{\xi^2}{4\eta_\infty}}.c_1 \tag{A.29}$$

Substituting the derivative of $\frac{dm}{d\xi}$ (Equation (A.29))into Equation (A.27) and integrating both sides of the resulting equation gives:

$$S_o(\xi) = c_1 \frac{N_i}{(m_i - m_w)} Erfc\left(\frac{\xi}{2\sqrt{\eta_\infty}}\right) + c_2 \tag{A.30}$$

where $c_1 = m_w - m_i = -(m_i - m_w)$

This yields;

$$S_o(\xi) = -N_i Erfc\left(\frac{\xi}{2\sqrt{\eta_\infty}}\right) + c_2 \tag{A.31}$$

Given that, the special value $Erfc(\infty) = 0$.

Evaluation $So(\xi)$ at $\xi = \infty$ shows that

$$So(\xi = \infty) = S_{oi} = 1 = c_2$$

Hence, the saturation solution becomes;

$$S_o(\xi) = 1 - N_i Erfc\left(\frac{\xi}{2\sqrt{\overline{\eta}_\infty}}\right) \tag{A.32}$$

Appendix B

Flowchart of the Algorithm for the Numerical Approach

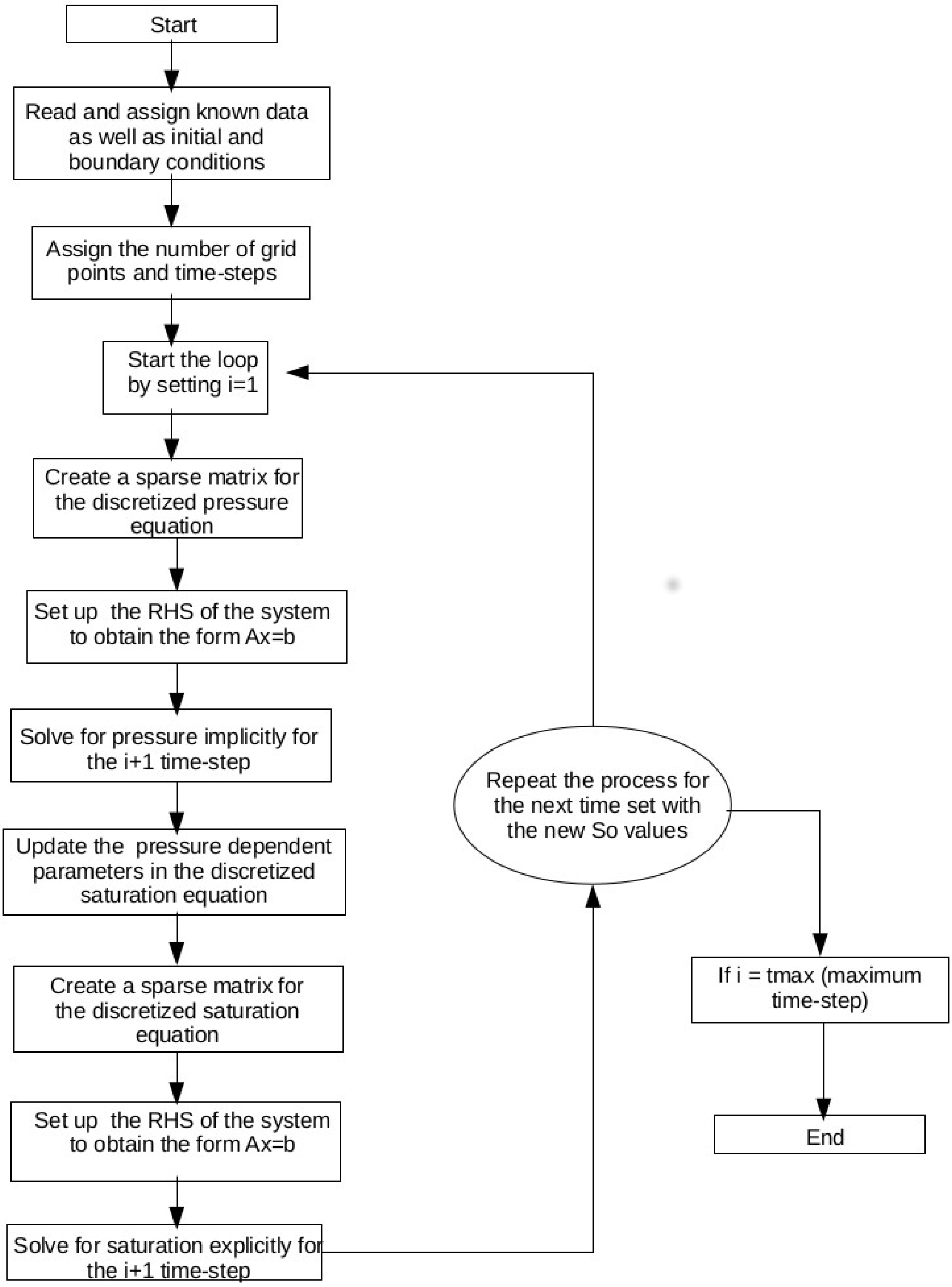

Figure B.1: A schematic drawing to illustrate the process involved in developing the Matlab code